Recitation 1: First Welfare Theorem

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1 Models

1. "Economists have chosen to abstract from the complexities of the real world and develop rather simple models that capture the essentials, just as a ***road map*** is helpful even though it does not record every house or every store." in Walter Nicholson's textbook

2. "The author of a ***fable*** draws a parallel to a situation in real life. He has some moral he wishes to impart to the reader. The fable is an imaginary situation that is somewhere between fantasy and reality. Any fable can be dismissed as being unrealistic or simplistic, but this is also the fable's advantage. Being something between fantasy and reality, a fable is free of extraneous details and annoying diversions. In this unencumbered state, we can clearly discern what cannot always be seen in the real world. On our return to reality, we are in possession of some sound advice or a relevant argument that can be used in the real world." in Rubinstein (2006 EMA)

3. "For some applications, a photograph may be the best means of ***depicting*** an object, but in some cases a drawing or even a caricature may allow greater understanding. Furthermore, a commercial artist who is attempting to provide an accurate impression of some object may utilize all three media, either sequentially or simultaneously. It is the same with the three techniques of economic modeling." Gibbard and Varian (1978)

4. "There cannot be a ***language*** more universal and more simple, more free from errors and obscurities, that is to say more worthy to express the invariable relations of natural things than mathematics." by Joseph Fourier, *Analytical Theory of Heat*, 1822

Remark 1. "All models are wrong but some are useful" (generally attributed to George Box). Models are abstractions, and they are intended to be abstractions of a much more complex reality, in order to be useful. For instance, you may not want to label Walnut street in a world map; a map of Philadelphia will not include the Pacific Ocean, even though it covers about one-third of Earth's total surface area. Modeling is the (unambiguous) language of economists that allows us to focus on one of the underlying mechanisms. Gilboa, Postlewaite, Samuelson and Schmeidler (2014) view models as case-based reasoning and hence one role of theory is to enrich the set of cases.

2 Walrasian Equilibrium

Definition 1. A Walrasian equilibrium (W.E.) for an exchange economy $E = \left(\left(u^h, e^h\right)_{h\in\mathcal{H}}\right)$ is a combination of prices and allocation $\left(p, \left(c^h\right)_{h\in\mathcal{H}}\right) \in \mathbb{R}^L \times \mathbb{R}^{HL}_+$ such that

1. Given prices p, each agent h solves $\max_{c\in\mathbb{R}^L_+} u^h(c)$ s.t. $p \cdot c \leq p \cdot e^h$
2. Markets clear: $\sum_{h\in\mathcal{H}} \left(c^h - e^h\right) = 0$

Remark 2. Equilibrium is of central importance in economics. This course first concerns Walrasian equilibrium (WE) in a pure exchange economy, and it is straightforward to be generalized to include firms (in a few lectures). Arrow and Debreu formalized Walras' model and hence it is often referred to as Arrow-Debreu equilibrium (ADE), or competitive equilibrium (CE). Later on, we will introduce "Debreu prices" to define ADE under uncertainty. We will also define Arrow security market equilibrium and establish the equivalence result. It will then be generalized to general equilibrium with incomplete markets (GEI). Macro courses often start from ADE and much stuff is (nontrivial) applications of general equilibrium theory; in addition, one will see recursive competitive equilibrium (RCE) to take advantage of dynamic programming tools. Game theory even talks about more equilibrium concepts: Nash equilibrium (NE), subgame perfect equilibrium (SPE), trembling hand perfect equilibrium (THP), Bayes-Nash equilibrium (BNE), weak perfect Bayesian equilibrium (WPBE), almost perfect Bayesian equilibrium (APBE), sequential equilibrium (SE), Markov Perfect Equilibria (MPE), etc.

So what is an equilibrium? An equilibrium concept is a mapping from environments (preferences, endowments, technology, information, etc.) to what will happen (allocations, prices, strategies, etc.) An equilibrium concept usually consists of optimality conditions plus some notion of consistency (sometimes with additional restrictions to make things more reasonable (and/or tractable), e.g. belief based refinements). For instance, in Definition 1, (1) is the optimality condition, and (2) captures a notion of consistency: prices are right in the sense that (mysteriously) markets should clear. As another example, Nash equilibrium requires: (1) each player is optimizing given beliefs about others' behavior; (2) these beliefs are correct in the sense that (mysteriously) beliefs are consistent with others' actual behavior.

Remark 3. The elements of a WE are allocations AND prices. Prices are the balance wheel of the market mechanism – prices serve as signals of relative supply and demand to buyers and sellers. The role of prices is to clear the markets. One big assumption of WE is that agents take prices as given. Later on we will see the game theoretic foundations of WE (Shapley and Shubik, 1977), where the NE outcome is that when agents are small relative to a market, their supply and demand will have a small impact on market price.

Remark 4. Although the definition for WE does not rule out negative prices, we could get rid of negative prices by assuming utility functions to be increasing (A2). Suppose the price for some good is negative, then there is always a way to buy more of every good and hence keep increasing your utility, without violating the budget constraint. Then there will be no well-defined solution to household optimization. But it is possible to have zero prices.

Remark 5. (A2) implies Walras' law, which in turn implies that one market clearing condition is redundant. That is, if we have market clearing in $L-1$ markets, then market clearing in the L-th market is spontaneously satisfied, which is guaranteed by Walras' law.

3 Pareto Optimal

Definition 2. A feasible allocation (i.e., nonnegative consumption that satises the resource constraint) $\left(x^h\right)_{h\in\mathcal{H}}$ is said to be a Pareto improvement for (or Pareto dominate) $\left(c^h\right)_{h\in\mathcal{H}}$ if

1. $u^h\left(x^h\right) \geq u^h\left(c^h\right), \forall h \in \mathcal{H}$
2. $u^{h'}\left(x^{h'}\right) > u^{h'}\left(c^{h'}\right), \exists h' \in \mathcal{H}$

Definition 3. Given an economy E, a feasible allocation $\left(c^h\right)_{h\in\mathcal{H}}$ is Pareto optimal (or Pareto efficient) if there is no other feasible allocation that Pareto dominates $\left(c^h\right)_{h\in\mathcal{H}}$.

Remark 6. Pareto optimality is achieved if there is no way to make one agent strictly better off without making someone else worse off. It has nothing to do with fairness in any sense. We could also define weak Pareto optimality if there is no other feasible allocation that makes everyone strictly better off. Pareto optimality implies weak Pareto optimality. If we assume utilities are increasing and continuous, then WPO also implies PO, i.e., they are equivalent: suppose $\left(c^h\right)_{h\in\mathcal{H}}$ is WPO but not PO, then $\exists\left(x^h\right)_{h\in\mathcal{H}}$ s.t. $u^h\left(x^h\right) \geq u^h\left(c^h\right), \forall h \in \mathcal{H}$ and $u^{h'}\left(x^{h'}\right) > u^{h'}\left(c^{h'}\right), \exists h' \in \mathcal{H}$. By continuity, we can take a little bit amount $\varepsilon \gg 0$ off from $x^{h'}$ but still leave him with $u^{h'}\left(x^{h'}-\varepsilon\right) > u^{h'}\left(c^{h'}\right)$, and redistribute ε evenly among others, then everyone is strictly better off $u^h\left(x^h + \frac{1}{H-1}\varepsilon\right) > u^h\left(x^h\right) \geq u^h\left(c^h\right)$, contradicting WPO.

Example 1. In an exchange economy, two agents have utility functions $u^i\left(x_1^i, x_2^i\right) = \min\left(x_1^i, x_2^i\right)$ and endowments $e^1 = (2,0)$ and $e^2 = (0,1)$.The Pareto set is[1] $\left\{x \in \mathbb{R}_+^4 \mid x^2 = (2,1) - x^1, x_1^i \geq x_2^i\right\}$. In the Edgeworth box, it is the area between the two 45 degree lines emanating from the agents' origins. In a competitive market, the equilibrium price is $p = (0,1)$, and agent 2 gets all of good 2 and at least as much of good 1; the equilibrium utilities are $(u_1, u_2) = (0,1)$.

[1] To be rigorous, one needs to prove it is indeed the Pareto set: for every allocation in it, making one strictly better off must worsen off someone else; for every allocation not in it, there is a Pareto improvement.

Now suppose that through some misfortune, 3/4 of agent 1's endowment is destroyed before trading occurs, leaving him with only $\hat{e}^1 = \left(\frac{1}{2}, 0\right)$. The new equilibrium price is $p = (1, 0)$, and agent 1 gets all of good 1 and at least as much of good 2; the new equilibrium utilities are $(u_1, u_2) = \left(\frac{1}{2}, 0\right)$. Agent 1 is made better off by a destruction of his endowment! In fact, one can show that if each agent is allowed to destroy a portion of his endowment before the competitive market opens, then the unique NE is such that they both destroy all their endowment.

Theorem 1. *Social Planner Characterization for Pareto Efficiency*

If $\left(x^h\right)_{h \in \mathcal{H}}$ solves the planner's problem for some vector of Pareto weights $\lambda \gg 0$

$$\left(x^h\right)_{h \in \mathcal{H}} \in \arg \max_{\left(c^h\right)_{h \in \mathcal{H}} \geq 0} \sum_h \lambda^h u^h \left(c^h\right)$$
$$s.t. \sum_h c^h \leq \sum_h e^h$$

then $\left(x^h\right)_{h \in \mathcal{H}}$ is Pareto efficient. Conversely, under weak assumption[2]*, any Pareto efficient allocation is a solution to the social planner problem for some weight vector $\lambda > 0$.*

Remark 7. This theorem immediately gives the first order condition characterizations of Pareto efficiency. If x is an *interior* Pareto optimal allocation, and all the utility functions have strictly positive partial derivatives at x, then social planner characterization implies that all agents have the same MRS at x. To see, define Lagrangian function $L = \sum_h \lambda^h u^h \left(c^h\right) + \sum_i \mu_i \left(\sum_h e_i^h - \sum_h c_i^h\right)$. Interior FOC gives $\forall h, \forall i$,

$$\frac{\partial L}{\partial c_i^h} = \lambda^h \frac{\partial u^h \left(c^h\right)}{\partial c_i^h} - \mu_i = 0$$

Therefore, $MRS_{i,j}^h = \frac{\partial u^h \left(c^h\right) / \partial c_i^h}{\partial u^h \left(c^h\right) / \partial c_j^h} = \frac{\mu_i}{\mu_j} = \frac{\partial u^{h'} \left(c^{h'}\right) / \partial c_i^{h'}}{\partial u^{h'} \left(c^{h'}\right) / \partial c_j^{h'}} = MRS_{i,j}^{h'}, \forall h, h', \forall i, j$. Note that the requirement of being interior is important here: the two origins of an Edgeworth box are indeed on the contract curve, but MRS need not be equalized at the origin.

The converse of is not true: equal MRS does not gurantee Pareto efficiency (draw a counterexample with nonconvex preferences in an Edgeworth box). If in addition to MRS being equal at x, all agents have convex preferences, then x is Pareto efficient.

4 First Welfare Theorem

Theorem 2. *First Welfare Theorem.*

[2]The assumption is if the utility possibility set is convex. See MWG Proposition 16.E.2.

Let $\left(p, \left(c^h\right)_{h\in\mathcal{H}}\right)$ *be a WE for the economy* E*. If (A2)*[3]*, then* $\left(c^h\right)_{h\in\mathcal{H}}$ *is PO.*

Proof. By contradiction. Suppose $\left(c^h\right)_{h\in\mathcal{H}}$ is not Pareto optimal: $\exists\left(x^h\right)_{h\in\mathcal{H}}$ feasible such that $u^h\left(x^h\right) \geq u^h\left(c^h\right), \forall h \in \mathcal{H}$ and $u^{h'}\left(x^{h'}\right) > u^{h'}\left(c^{h'}\right), \exists h' \in \mathcal{H}$.

Claim 1. $p \cdot x^{h'} > p \cdot c^{h'}$. ($p \cdot x^{h'} \leq p \cdot c^{h'}$ contradicts to $c^{h'}$ being the optimal choice by h'.)

Claim 2. $p \cdot x^h \geq p \cdot c^h, \forall h$. (If $u^h\left(x^h\right) > u^h\left(c^h\right)$, by Claim 1 we have $p \cdot x^h > p \cdot c^h$. If $u^h\left(x^h\right) = u^h\left(c^h\right)$, suppose $p \cdot x^h < p \cdot c^h$, then by (A2) $\exists x'$ s.t. $p \cdot x' \leq p \cdot c^h$ but $u^h\left(x'\right) > u^h\left(c^h\right)$.)

The above two claims imply $\sum_{h=1}^H p \cdot x^h > \sum_{h=1}^H p \cdot c^h$. This implies $\exists l$ s.t. $\sum_{h=1}^H p_l \cdot x_l^h > \sum_{h=1}^H p_l \cdot c_l^h$. Since prices are nonnegative, $\sum_{h=1}^H x_l^h > \sum_{h=1}^H c_l^h = \sum_{h=1}^H e_l^h$ violates feasibility. $\square$

Remark 8. A simple but very sloppy "proof" (which is not really a proof but an exposition of the relation between WE and PO): If $\left(c^h\right)_{h\in\mathcal{H}}$ is an interior WE allocation, and if $u^h \in C^1, \forall h$, then individual optimality implies $MRS_{xy}^h = \frac{p_x}{p_y}$. Since prices are the same for everyone, then MRS are equalized among households $MRS_{xy}^h = MRS_{xy}^{h'}, \forall h, h' \in \mathcal{H}$, which is consistent with PO. (See Remark 7)

Remark 9. One application of FWT and Theorem 1 is Negishi's (1960) method which simplies computing Walrasian equilibria. The main idea is to first compute Pareto optimal allocations by solving social planner's problem, giving all potential equilibrium allocations according to FWT. Then isolate the ones that are indeed Walrasian equilibrium allocations, by picking the right Pareto weights that make the transfer functions to be 0.

Example 2. Two consumers have utility functions $u^i\left(x_1^i, x_2^i\right) = \ln\left(x_1^i\right) + \ln\left(x_2^i\right)$ and endowments $\omega^1 = (3, 1), \omega^2 = (1, 1)$. The government is considering two policies:

A. Tax/transfer, then trade: The government requires consumer 1 to transfer 1 unit of good 1 to consumer 2. Then consumers trade according to the competitive equilibrium.

B. Trade, then tax/transfer: Consumers trade according to the competitive equilibrium. Then the government requires consumer 1 to transfer 1 unit of good 1 to consumer 2.

The first welfare theorem tells us that policy A is Pareto optimal. Generically, an arbitrary reallocation of WE (policy B) will not be Pareto optimal. What is more, in this setting, simple calculation gives that Policy A actually Pareto dominates Policy B: the final allocation is $((2, 1), (2, 1))$ under policy A, and is $\left(\left(\frac{3}{2}, \frac{5}{4}\right), \left(\frac{5}{2}, \frac{3}{4}\right)\right)$ under policy B.

Remark 10. The First Welfare Theorem suggests that the market is an efficient organization, which formalizes Adam Smith's invisible hand. But also pay attention to the (both explicit

[3] We will frequently refer to the following assumptions: For all agents $h \in \mathcal{H}$, (A1) $e^h \gg 0$; (A2) utility is increasing: $\forall x, y \in \mathbb{R}_+^L$, $u^h(x) > u^h(y)$ whenever $x \gg y$; (A3) $u^h(\cdot)$ is continuous; (A4) $u^h(\cdot)$ is concave.

and implicit) assumptions we have. For example, when agents do not take prices as given due to market power (i.e., some consumers are acting like monopolists); or if we do not have a complete market (e.g. externality), Pareto efficiency may be violated. We will talk about more circumstances about market failures (asymmetric information, public goods, etc.) at the end of this course. Here are some examples showing failures of the first welfare theorem.

Example 3. OLG. Suppose the endowments are $e_1^0 = 0$, $\left(e_t^t, e_{t+1}^t\right) = (1, 0), \forall t \geq 1$. Utility function of generation t is $u^t\left(c_t^t, c_{t+1}^t\right) = c_t^t + c_{t+1}^t$. Autarky is WE but not Pareto optimal. The allocation $x_1^0 = \frac{1}{2}$, $\left(e_t^t, e_{t+1}^t\right) = \left(\frac{1}{2^t}, 1 - \frac{1}{2^{t+1}}\right), \forall t \geq 1$ is feasible and and Pareto dominates autarky allocation. It looks like a Ponzi scheme but since we assume the economy lasts forever and there are infinite many generations, it is indeed feasible.

Why FWT fails in this example? Note that in our environment, we have assumed $|\mathcal{H}| = H$ and $|\mathcal{L}| = L$, i.e., finite households and finite commodities. In fact, FWT can be generalized to infinite dimensional commodity space. However, in the OLG model, we have both an infinite number of periods (hence infinite number of commodities) as well as an infinite number of agents. This double infinity is crucial for the failure of the first welfare theorem: the infinite sum that appears in the proof may be ill-defined. Double infinity is the major source of the theoretical peculiarities of OLG models (e.g. Karl Shell, 1971, JPE).

Example 4. Missing Market. Suppose there are two agents with strictly increasing utility[4] and two commodities, but they cannot trade commodity 2 (i.e., missing market for good 2). Typically Walrasian equilibria are not Pareto efficient here (since the Walrasian equilibrium will be the endowment).

Example 5. Externality. Suppose two households have utility functions $u^1(x) = \ln(x_1^1 + x_1^2) + \ln(x_2^1)$ and $u^2(x) = \ln x_1^2 + \ln x_2^2$. There is externality in the sense that 2's consumption enters 1's utility. The set of Pareto optimal allocations is

$$PO = \left\{x \in \mathbb{R}_+^4 \mid x_1^1 = 0, x_1^2 = e_1, x_2^1 + x_2^2 = e_2\right\}$$

Suppose the endowment is $e^1 = (1, 2)$ and $e^2 = (2, 1)$. The Walrasian equilibrium is (a calculation exercise) $\frac{p_2}{p_1} = \frac{8}{5}$ and $x^1 = \left(\frac{6}{5}, \frac{15}{8}\right)$, $x^2 = \left(\frac{9}{5}, \frac{9}{8}\right)$, which is not in PO. (Be careful that it is a big mistake to substitute market clearing conditions into the individual's problem before maximization.) Externality can also be viewed as a "missing market" in the sense that the bystander neither pays nor gets compensated for what affects her utility.

[4] A nice property of utility function is that it guarantees the equilibrium price $p \gg 0$ (if equilibrium exists). Then we do not need to discuss zero prices, and can normalize one price to be 1. In fact, if someone's utility function is strictly in consumption in good i, then p_i must be positive; otherwise he will demand infinite amount of good i.

Recitation 2: Second Welfare Theorem

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1 Second Welfare Theorem

Theorem 1. *(Second Welfare Theorem) An economy E satisfies (A1)-(A4). If is $\left(e^h\right)_{h\in\mathcal{H}} \gg 0$ is Pareto-efficient, then $\exists p \in \mathbb{R}_+^L$ such that $\left(p, \left(e^h\right)_{h\in\mathcal{H}}\right)$ is a Walrasian Equilibrium for E.*

Remark 1. The First Welfare Theorem says that (under the assumptions) Walrasian Equilibrium allocations are Pareto Efficient. The Second Welfare Theorem gives the converse: every Pareto Efficient allocation can be obtained as a Walrasian Equilibrium allocation if we allow for a redistribution of endowment (and the above statement proves it by finding a particular way of reallocation). Pareto efficiency is a weak result as it does not say anything about a "just" distribution: an economy where the wealthy hold the vast majority of resources could still be Pareto efficient but might not be desirable. (But this weakness does not mean that we will give up the idea of Pareto efficiency; instead, it should be viewed as the minimal requirement that we will pursue.) Though FWT says markets are good in that WE are PE, it is silent about the desirability from a distributional perspective. SWT complements FWT in the sense that it gives conditions under which any desired efficient allocation could be decentralized through competitive markets, as long as reallocation is possible. (However, bear in mind that in reality SWT seldom works due to the government's inability to enact the lump-sum transfer.)

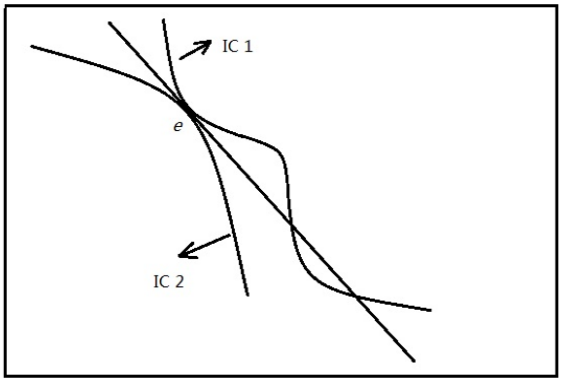
Proof. The idea of the proof is the following (for more details please see the notes). First we construct a preferred set $K^h = \left\{z \in \mathbb{R}^L : e^h + z \geq 0 \text{ and } u^h\left(e^h + z\right) > u^h\left(e^h\right)\right\}$ for every agent. Let $K = \sum_h K^h$. By the concavity of utility functions, K is convex. Since $\left(e^h\right)_{h\in\mathcal{H}}$ is Pareto efficient, $0 \notin K$. Therefore by separating hyperplane theorem, there exists $p \neq 0$ such that $p \cdot z \geq p \cdot 0 = 0, \forall z \in \text{cl}(K)$. We can proceed to show under this p, no trade is an equilibrium. □

Theorem 2. *(Separating Hyperplane Theorem) Suppose that $B \subset \mathbb{R}^L$ is convex and $x \notin int(B)$. Then $\exists p \in \mathbb{R}^L, p \neq 0$ such that $p \cdot y \geq p \cdot x, \forall y \in B$.*

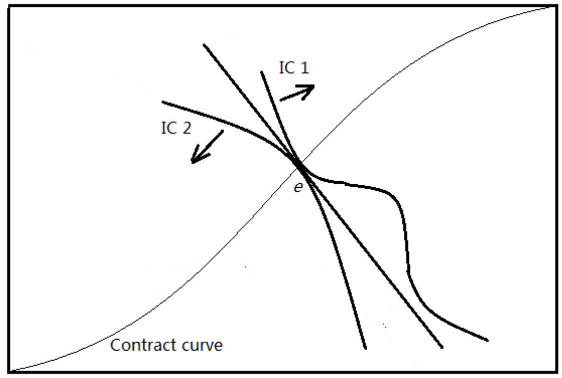
Example 1. Foley (1967) proposed a notion of "fairness" where no one covets another's allocation. Formally, an allocation $\left\{x^h\right\}_{h\in\mathcal{H}}$ is no-envy if $u^h\left(x^h\right) \geq u^h\left(x^{h'}\right), \forall h, h'$ (akin

Figure 1: Nonconvexity

(a) SWF Fails with Nonconvexity



(b) SWF Holds with Nonconvexity



to incentive compatibility). If one pursues an outcome that is both Pareto-efficient and no-envy, it can be achieved through a market mechanism by an egalitarian reallocation. Let $\hat{e}^h = \frac{1}{H}\sum_{i=1}^{H} e^i, \forall h$. By FWT, the equilibrium allocation $\left\{\hat{x}^h\right\}_{h\in\mathcal{H}}$ under this reallocation (as long as it exists) is Pareto-efficient. To see that $\left\{\hat{x}^h\right\}_{h\in\mathcal{H}}$ is also no-envy, notice that Walras' law implies $p\cdot\hat{x}^h = p\cdot\hat{x}^{h'} = p\cdot\frac{\sum_i e^i}{H}, \forall h, h'$. By the revealed preference argument, $u^h\left(\hat{x}^h\right) \geq u^h\left(\hat{x}^{h'}\right), \forall h, h'$.

Example 2. Figure 1a gives an example where assumptions of SWT are not satisfied and the conclusion of SWT fails. Suppose the two indifference curves are tangent at e. The allocation e is Pareto efficient. However it can not be an equilibrium allocation because given the price vector, agent 1 is not optimizing.

Example 3. Figure 1b gives an example where assumptions of SWT are not satisfied but the conclusion of SWT holds. If at *every* point on the contract curve, the two indifference curves are tangent as in the figure, then the conclusion of the second welfare theorem holds even if the preference is not convex. Notice it is not enough if only one Pareto efficient allocation is supported as an equilibrium because second welfare theorem says all Pareto efficient allocations can be supported as equilibria.

Remark 2. If the existence of WE is already established, then SWT is immediate. Start with an economy where the endowments $\left(e^h\right)_{h\in\mathcal{H}}$ are Pareto-efficient and suppose $\left(c^h\right)_{h\in\mathcal{H}}$ is an equilibrium allocation. People trade to $\left(c^h\right)_{h\in\mathcal{H}}$ only if $u\left(c^h\right) \geq u\left(e^h\right), \forall h$. If there is a strict inequality, then Pareto efficiency of $\left(e^h\right)_{h\in\mathcal{H}}$ is violated. So there is no strict inequality, which implies $\left(e^h\right)_{h\in\mathcal{H}}$ must be an equilibrium allocation. However, as we will see below, the proof of existence invokes the fixed point theorem (which is harder), while the above proof for SWT only uses the separating hyperplane theorem (which is easier).

2 Existence

Theorem 3. *(Existence) A Walrasian equilibrium exists for a pure exchange economy if each agent has (A1) positive endowments, utility functions being (A2') strictly increasing, (A3) continuous, (A4') strictly concave.*

Example 4. Conditions in the above theorem are sufficient but not necessary. Consider two agents with Leontief preferences $u^i(x_1^i, x_2^i) = \min\{x_1^i, x_2^i\}$ and endowments $e^1 = (1,0), e^2 = (0,1)$. There exist multiple WE where prices (p_1, p_2) can range all from $(1,0)$ to $(0,1)$. Even if the utility functions are $u^i(x_1^i, x_2^i) = \max\{x_1^i, x_2^i\}$ which is not convex, two WE exist: one is $p = (1,1)$ and $x = ((1,0),(0,1))$, the other is $p = (1,1)$ and $x = ((0,1),(1,0))$.

Remark 3. Note that (A4') + (A2) $\Rightarrow$ (A2'). If you are really ambitious, a more general existence theorem can be found in Chapter 5.3 in Bewley. There we invoke the Kakutani fixed point theorem, which generalizes the Brouwer's theorem to correspondences.

Definition 1. Aggregate excess demand is defined as $z(p) = \sum_h \left(f^h\left(p, e^h\right) - e^h\right)$, where f^h is the individual demand function.

Remark 4. Some things to note:

- As seen in 701A, demand functions are homogeneous of degree 0 and thus we can normalize the prices. It seems attractive is to normalize one good to be numeraire ($p_1 = 1$), but this requires p_1 positive before normalization. To get rid of this problem, we could instead normalize $p \in \Delta^{L-1}$.
- Again in 701A we have seen that (A2) implies Walras' law: $p \cdot f^h\left(p, e^h\right) = p \cdot e^h$ and hence $p \cdot z(p) = 0, \forall p$ (not only at equilibrium prices).
- p^* is an equilibrium price if and only if $z(p^*) = 0$.

Theorem 4. *(Brouwer's fixed point theorem) Given a compact convex set $A \subset \mathbb{R}^n$ and a continuous function $f : A \to A$, there exists $x \in A$ such that $f(x) = x$.*

Remark 5. Examples of $f : A \to A$ with some assumption violated, and f has no fixed point:

1. (If A is not closed) $A = (0,1)$, and $f(x) = x^2$.
2. (If A is not bounded) $A = [0, +\infty)$, and $f(x) = x + 1$.
3. (If A is not convex) $A = \{x \in \mathbb{R}^2 : \|x\| = 1\}$, and $f(x) = -x$.
4. (If f is not continuous) $A = [0,1]$, and $f(x) = 1\{x \leq 0.5\}$.

Theorem 5. *Suppose the aggregate excess demand function $z : \Delta^{L-1} \to \mathbb{R}^L$ is continuous, satisfies Walras' law and is homogeneous of degree 0. Then there exist a p^* such that $z(p^*) \leq 0$ and $z_l(p^*) = 0$ if $p_l^* > 0$.*

Proof. The idea is to construct a function $g : \Delta^{L-1} \to \Delta^{L-1}$ by

$$g(\bar{p}) = \arg\max_{p \in \Delta^{L-1}} \left(pz(\bar{p}) - \|p - \bar{p}\|^2 \right)$$

Then we can argue that g has a fixed point p^* by verifying conditions in Brouwer's theorem. Lastly we can show $p \cdot z(p^*) \leq 0, \forall p \in \Delta^{L-1}$ by contradiction, using $p_\varepsilon = \varepsilon p + (1-\varepsilon) p^*$. $\square$

Remark 6. How does this result relate to the existence of Walrasian equilibrium? First, optimality conditions are already incorporated in demand functions. Second, notice that, the result already gives the existence of a slightly different definition of "equilibrium": for $p_l^* > 0$, we have exact market clearing $z_l(p^*) = 0$; for $p_l^* = 0$, we allow for demand to be less than endowments, i.e, $z_l(p^*) \leq 0$. However, if we assume utility functions are strictly increasing, then in order to get well-defined demand, prices have to be strictly positive. That is, we can get rid of this problem and get back to exact market clearing $z(p^*) = 0$.

Example 5. (Boundary endowments) Consider an economy with 2 agents and 3 commodities. The endowments are $e^1 = (1,1,1)$, $e^2 = (0,2,0)$. Suppose $u^1(x_1, x_2, x_3) = \sqrt{x_1} + x_3$, $u^2(x_1, x_2, x_3) = x_1 + x_2$. Strict monotonicity implies the equilibrium prices must be positive if an equilibrium exists. Then agent 1 will sell good 2 and has excess demand for good 1 and/or 3, but markets cannot clear since agent 2 has none of them.

Example 6. (Non Convexity) Endowments are $e^1 = (1,0), e^2 = (0,1)$. Utility functions are $u^1(x_1, x_2) = (x_1)^2 + (x_2)^2$, $u^2(x_1, x_2) = x_1 x_2$. At any strictly positive price vector, agent 1's optimal bundle is a boundary point while agent 2's is interior. There cannot be an equilibrium with a price of 0 for either good since both agents have strictly monotonic preferences for both goods.

Example 7. Figure 1a also gives an example where assumptions of the existence theorem are not satisfied and equilibrium does not exist. Figure 1b also gives an example where assumptions of the existence theorem are not satisfied and but equilibrium exists. (Recall in Remark 2 we talk about the relation of existence and SWT.)

A More Results

We have some other interesting results but we will not pursue too much in this course:

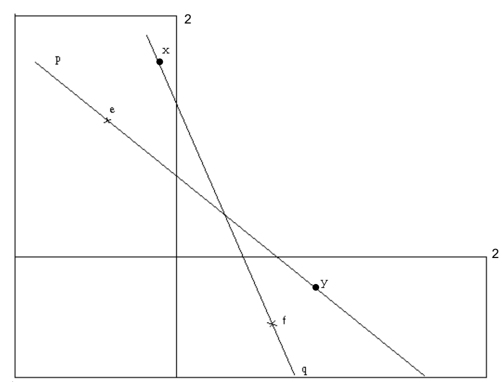
1. Global uniqueness: Mitiushin and Polterovich provides nice sufficient conditions to ensure uniqueness.

2. Tatonnement stability: After Scarf's (1960) example with unique equilibrium where the tatonnement process never converges, this idea became less popular. Rational expectation interpretation of WE: agents correctly anticipate the correct prices, then choose according to these prices and then the prices turn out to be correct.

3. Sonnenschein, Mantel, Debreu (SMD): "Anything goes" with an excess demand function (as long as homogeneity and Walras' law are satisfied). The implication is one can have an arbitrary number of equilibria with arbitrary stability properties.

4. The equilibrium correspondence is upper-hemi continuous. Global restrictions on equilibrium correspondence: not "anything goes" in general equilibrium theory (Brown and Matzkin, 1996)

Figure 2: Impossibility of Equilibrium Correspondence



Notes: Cyclical consistency is violated for agent 1, for example, $y \succ_1 f \succ_1 x \succ_1 e \succ_1 y$.

Recitation 3: Production

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1 Production Economy

We now generalize the Walrasian model of a pure exchange economy to include production: K firms with production sets $Y^k \subset \mathbb{R}^L$,and share δ_k^h owned household h. A production economy is therefore denoted by $E = \left(\left(u^h, e^h, \left(\delta_k^h\right)_{k\in\mathcal{K}}\right)_{h\in\mathcal{H}}, \left(Y^k\right)_{k\in\mathcal{K}}\right)$.

Definition 1. WE for a production economy E is defined as $\left(p, \left(c^h\right)_{h\in\mathcal{H}}, \left(y^k\right)_{k\in\mathcal{K}}\right)$ s.t.

1. Given prices, households optimize: $c^h \in \arg\max_{c\in\mathbb{R}_+^L} u^h(c)$ s.t. $p\cdot c \leq p\cdot e^h + p\cdot\sum_k \delta_k^h y^k$
2. Given prices, firms maximize profits: $y^k \in \arg\max_{y\in Y^k} p\cdot y$
3. Markets clear: $\sum_h c^h = \sum_h e^h + \sum_k y^k$

It is straightforward to extend the first welfare theorem, the second welfare theorem, and existence obtained in a pure exchange economy to a production economy.

Example 1. Consider a two person economy in which each person has an endowment of 2 units of l. Agent 1 owns the production function for good x, which is $x = 2\sqrt{l}$, and agent 2 owns the production function for good y, which is $y = 2\sqrt{l}$. Agents utility functions are: $u^1(l,x,y) = l + x + y, u^2(l,x,y) = lx^2y^3$. A first observation is that the equilibrium prices must be strictly positive, since u^1 is strictly increasing in each argument. We can thus normalize $p_l = 1$. Firm 1's profit maximization gives $l_x = p_x^2$, $x = 2p_x$ and $\pi_x = p_x^2$. The solution for firm 2 is similar, by replacing subscripts with y.

Let's try an interior equilibrium. Then it must be $p_l = p_x = p_y = 1$ due to agent 1's utility function. Therefore, the wealth for two agents are $m^1 = 2+\pi_x = 3$ and $m^2 = 2+\pi_y = 3$. The Cobb-Douglas form then immediately gives agent 2's demand: $(l_2, x_2, y_2) = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)\times 3 = (0.5, 1, 1.5)$. Then by market clearing $(l_1, x_1, y_1) = (1.5, 1, 0.5)$, which is indeed optimal for agent 1 because they exhaust her wealth. So we find a WE.

Theorem 1. *(First Welfare Theorem) Assume (A2). Any Walrasian Equilibrium allocation for a production economy is Pareto efficient.*[1]

[1]The passage in *The Wealth of Nations* deserves to be repeated here: "As every individual, therefore, endeavours as much as he can both to employ his capital in the support of domestic industry, and so to

Proof. The proof is by contradiction. If $\left(\left(c^h\right)_{h\in\mathcal{H}}, \left(y^k\right)_{k\in\mathcal{K}}\right)$ is a Walrasian Equilibrium but not Pareto efficient, then there exists a feasible allocation $\left(\left(\hat{c}^h\right)_{h\in\mathcal{H}}, \left(\hat{y}^k\right)_{k\in\mathcal{K}}\right)$ that Pareto dominates it. By the same argument as before, $p \cdot \hat{c}^{h'} > p \cdot c^{h'}$ for some h' and $p \cdot \hat{c}^h \geq p \cdot c^h$ for any h, and hence $\sum_h p \cdot \hat{c}^h > \sum_h p \cdot c^h$. This is equivalent to $p \cdot \sum_h \hat{c}^h > p \cdot \sum_h c^h$. (1)

Firms' profit maximization implies $p \cdot y^k \geq p \cdot \hat{y}^k, \forall k$, so $\sum_k p \cdot y^k \geq \sum_k p \cdot \hat{y}^k$. This is equivalent to $p \cdot \sum_k y^k \geq p \cdot \sum_k \hat{y}^k$. (2)

(1) and (2) together imply $p \cdot \left(\sum_h \hat{c}^h - \sum_k \hat{y}^k\right) > p \cdot \left(\sum_h c^h - \sum_k y^k\right) = p \cdot \sum_h e^h$. The last equality comes from market clearing condition. Since equilibrium prices are nonnegative under (A2), this means $\exists l$ s.t. $\sum_h \hat{c}_l^h - \sum_k \hat{y}_l^k > \sum_h e_l^h$, which violates feasibility. □

Remark 1. Assuming differentiability, interior Pareto efficient allocations and production plans for a production economy with a production function $y = f(x)$ can be characterized by $MRS^h = MRS^{h'} = MRT$.

For the second welfare theorem, the key idea is: the price vector is nothing but a hyperplane that separates the aggregate post-production set from the aggregate strictly preferred set. We need another version of separating hyperplane theorem to separate two convex sets (instead of separating a convex set and a point as we did before). Thus we also need convexity of production sets.

Assumption. *(A5) Y^k is closed and convex.*

Lemma 1. *(Convex Separating Theorem) Let $D, E \subseteq \mathbb{R}^L$ be disjoint, nonempty, convex.*

- *$\exists H(p, a)$ such that $p \cdot d \geq a \geq p \cdot e$, $\forall d \in D, \forall e \in E$.*
- *If D is open, then $\exists H(p, a)$ such that $p \cdot d > a \geq p \cdot e$, $\forall d \in D, \forall e \in E$.*
- *If both are open, then $\exists H(p, a)$ such that $p \cdot d > a > p \cdot e$, $\forall d \in D, \forall e \in E$.*

Theorem 2. *(Second Welfare Theorem) Assume (A2)-(A5). Every interior Pareto efficient allocation can be decentralized in Walrasian equilibrium.*

direct that industry that its produce may be of the greatest value; every individual necessarily labours to render the annual revenue of the society as great as he can. He generally, indeed, neither intends to promote the public interest, nor knows how much he is promoting it. By preferring the support of domestic to that of foreign industry, he intends only his own security; and by directing that industry in such a manner as its produce may be of the greatest value, *he intends only his own gain, and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention.* Nor is it always the worse for the society that it was no part of it. By pursuing his own interest he frequently promotes that of the society more effectually than when he really intends to promote it. I have never known much good done by those who affected to trade for the public good. It is an affectation, indeed, not very common among merchants, and very few words need be employed in dissuading them from it."

Proof. Consider an interior Pareto efficient allocation $\left(\left(c^h\right)_{h\in\mathcal{H}}, \left(y^k\right)_{k\in\mathcal{K}}\right)$ with $c^h \gg 0$. WTS there is a distribution of the economy's resources $\left(\left(e^h, \left(\delta_k^h\right)_{k\in\mathcal{K}}\right)_{h\in\mathcal{H}}\right)$ and a price vector p such that $\left(p, \left(c^h\right)_{h\in\mathcal{H}}, \left(y^k\right)_{k\in\mathcal{K}}\right)$ is a WE.

Define $K^h = \left\{z \in \mathbb{R}^L : u^h(z) > u^h\left(c^h\right)\right\}$ and $K = \sum_h K^h$ ("aggregate strictly preferred set"). Let $W = e + \sum_k Y^k$ ("aggregate post-production set"), where e is the aggregate endowment. Under the assumptions, K is open, K and W are convex. Moreover, they are disjoint; otherwise Pareto efficiency is violated. By the convex separating theorem, there exists p such that $p \cdot z > p \cdot w, \forall z \in K, \forall w \in W$.

We now verify that this price p indeed supports $\left(\left(c^h\right)_{h\in\mathcal{H}}, \left(y^k\right)_{k\in\mathcal{K}}\right)$ as a WE. The inequality obtained above can be translated to that $\forall \left(\left(\hat{c}^h\right)_{h\in\mathcal{H}}, \left(\hat{y}^k\right)_{k\in\mathcal{K}}\right)$ such that $u^h\left(\hat{c}^h\right) > u^h\left(c^h\right), \forall h$ and $\hat{y}^k \in Y^k, \forall k$, we have $p \cdot \sum_h \hat{c}^h > p \cdot e + p \cdot \sum_k \hat{y}^k$. Since $\sum_h c^h = e + \sum_k y^k$, it can be written as

$$p \cdot \sum_h \left(\hat{c}^h - c^h\right) > p \cdot \sum_k \left(\hat{y}^k - y^k\right)$$

First, take a sequence such that $\hat{c}^h \to c^h, \forall h$, and $\hat{y}^k = y^k, \forall k \neq j$, then in the limit we get $p \cdot (\hat{y}^j - y^j) \leq 0$, which proves that y^j maximizes profit given p. This is true for arbitrary $j \in \mathcal{K}$.

Second, take a sequence such that $\hat{y}^k = y^k, \forall k$, and $\hat{c}^h \to c^h, \forall h \neq i$, then in the limit we get $u^i(\hat{c}^i) > u^i(c^i) \Rightarrow p \cdot (\hat{c}^i - c^i) \geq 0$. This is true for arbitrary $i \in \mathcal{H}$. Then we show it can never be an exact equality. Suppose $u^i(\hat{c}^i) > u^i(c^i)$ and $p \cdot \hat{c}^i = p \cdot c^i$, then by continuity $\exists \lambda < 1$ such that $u^i(\lambda\hat{c}^i) > u^i(c^i)$ and $p \cdot (\lambda\hat{c}^i) < p \cdot c^i$, which is a contradiction. Therefore,

$$u^i\left(\hat{c}^i\right) > u^i\left(c^i\right) \Rightarrow p \cdot \hat{c}^i > p \cdot c^i, \forall i \in \mathcal{H}$$

which means that for every i, c^i maximizes household i's utility under budget constraint. □

Existence can be obtained in a production economy as well, *mutatis mutandis.* In particular, we need to have the supply correspondences in the aggregate excess demand. To invoke the fixed point theorem, we want to ensure that supply correspondences are non-empty, convex-valued and uhc in prices. Thus we need more assumption on production sets.

Assumption. *(A6)* $0 \in Y^k$ *(possibility of inaction);* $\mathbb{R}^L_{--} \subset Y^k$ *(free disposability).*

Assumption. *(A7) Denote* $Y = \sum_k Y^k$*, then* $Y \cap -Y = \{0\}$.

Remark 2. (A7) assumes that production is irreversible. If $y \in Y$ and $y \in -Y$, then $-y \in Y$, i.e., $-y$ is also a possible production vector. y is the reverse of $-y$ in the sense that the outputs in y are the inputs in $-y$ and vice versa.

Theorem 3. *(Existence) With (A5)-(A7) in addition, existence of Walrasian equilibrium carries over to the case of a production economy.*

Proof. For a complete treatment, please refer to Bewley, Section 4.8, which is in turn based on Debreu (1959), Section 7 of Chapter 5. The idea of proof is the following. Now we modify the aggregate excess demand to be $z(p) = \sum_h \left(f^h\left(p, e^h\right) - e^h\right) - \sum_k y^k(p)$, where $y^k(p)$ is the supply correspondence for firm k. $z(p)$ is homogeneous of degree 0 in p.

First, we can show that the set of feasible allocations is compact and nonempty (See Bewley Theorem 3.54). One can thus place a limit on the amount of any commodity available to any firm or consumer, and the resulting truncated production sets and budget sets are compact. Therefore, the demand and supply defined over them are uhc, convex valued and non-empty. Then we construct the same mapping for the existence proof for exchange economies (but with the excess demand adding supply correspondence), and by the same argument there will be a fixed point. Finally, verify that this fixed point is indeed an equilibrium price vector for the production economy with no truncation. □

We do not provide more examples about existence and non-existence, because it would replicate Kim Border's amazing notes on "(Non)-Existence of Walrasian Equilibrium."

http://www.its.caltech.edu/~kcborder/Notes/Walrasian.pdf

2 Linear Technology

A simple example of production technology is the "linear activity model", which assumes production set is the convex cone spanned by finitely many rays:

$$Y = \left\{ y \in \mathbb{R}^L : y = \sum_{m=1}^{M} \gamma_m a_m, \text{ for some } \gamma \in \mathbb{R}_+^M \right\}$$

where a_m is some linear activity, and $\gamma_m \geq 0$ is the amount of a_m that is used in production.

Remark 3. Do not confuse this formula with convex combination: there is NO requirement that $\sum_{m=1}^{M} \gamma_m = 1$. We do require $\gamma \geq 0$. The interpretation of γ_m is levels, not weights.

Remark 4. The linear technology Y defined above exhibits constant returns to scale (CRS): $y \in Y \Rightarrow \lambda y \in Y, \forall \lambda \geq 0$. Some might have a feeling that CRS is too strong an assumption and that in reality technologies are often decreasing return to scale: if you duplicate a factory by doubling all the inputs, you will probably get less than twice the output. A compelling reason is that managerial skill is the constraint. But this is saying that managerial skill, as an input, is fixed. This idea can be summarized as below: non-increasing returns to scale technology can be thought of as a constant-returns-to-scale technology with some input fixed.

Formally, for any convex production set $Y \subset \mathbb{R}^L$ with $0 \in Y$, there is a constant-returns-to-scale convex production set $Y' \subset \mathbb{R}^{L+1}$ such that $Y = \left\{y \in \mathbb{R}^L : (y, -1) \in Y'\right\}$. To see, let

$$Y' = \left\{y' \in \mathbb{R}^{L+1} : y' = \alpha\,(y, -1) \text{ for some } y \in Y, \alpha \geq 0\right\}$$

If Y is convex, so is Y'. If $y' \in Y'$, so is $\lambda y', \forall \lambda \geq 0$, hence Y' is CRS. It is straightforward that $Y = \left\{y \in \mathbb{R}^L : (y, -1) \in Y'\right\}$. This provides a justification for linear technology models.

Remark 5. A profit maximizing production plan is well defined if and only if $p \cdot a_m \leq 0, \forall m$. If $p \cdot a_m > 0$ for some m, the firm will choose $\gamma_m \to \infty$ and make infinite profits. If $p \cdot a_m < 0$ for some m, the firm will not operate this activity, i.e., $\gamma_m = 0$. If $p \cdot a_m = 0$ for some m, the firm may or may not use this activity. But if the firm is using activity a_m, then it must be that $p \cdot a_m = 0$ (zero profit conditions). This observation already tells a lot about equilibrium prices. The levels could be pinned down by market clearing conditions in equilibrium.

Remark 6. Under linear technology, profits are always 0 in equilibrium, so firms do not play a role at all, and it makes no difference who owns production. Actually this is true more generally, even for strictly convex production sets. That is, W.E. with firms is equivalent to a household production equilibrium.

Example 2. Consider a country with two tradeable commodities, plastic and oil, and a third commodity, pollution, that is not priced or traded. There is a single firm with a CRS technology $Y = \{(\gamma, -\gamma, \gamma) : \gamma \geq 0\}$. There is a single consumer with endowment $e_2 = 1$ and utility $u\,(x_1, x_2, y_3) = \ln x_1 + \ln x_2 - \frac{3}{2} y_3$.

Normalize $p_2 = 1$ so oil is the numeraire. The linear technology implies zero profit, and hence $p_1 = p_2 = 1$. Household's optimal consumption is thus $x_1 = \frac{1}{2}, x_2 = \frac{1}{2}$. In the Walrasian equilibrium, market clearing pins down that the firm converts $\frac{1}{2}$ unit of oil into plastic, and generates $\frac{1}{2}$ unit of pollution.

This is not Pareto optimal, however, due to presence of externality (or "missing market"). The Pareto efficient allocation solves: $\max_{x_1} \ln x_1 + \ln (1 - x_1) - \frac{3}{2} x_1$, which gives $x_1^* = \frac{1}{3}$. There is over-production under negative externality.

The government can decentralize the Pareto efficient allocation by choosing a suitable tax τ on plastic and redistributing the proceeds to the consumer as a lump-sum transfer T. Under this scheme, $p_1 = 1 + \tau$ and hence Household's optimal consumption is $x_1 = \frac{1+T}{2(1+\tau)}, x_2 = \frac{1+T}{2}$. We can achieve Pareto efficiency if $\tau = 1$.

Example 3. Consider a production economy with 2 agents and 4 commodities. Commodities 1 and 2 are consumption goods while commodities 3 and 4 are production inputs. Utility functions are $u^i(c_1, c_2, c_3, c_4) = \ln c_1 + \ln c_2, \forall i = 1, 2$. Endowments are $e^1 = (0, 0, 1, 0), e^2 = (0, 0, 0, 2)$. Suppose there are 4 possible activities transforming inputs into commodities 1 and 2: $a_1 = (1, 0, -3, 0), a_2 = (0.1, 1, -1, 0), a_3 = (1, 0, 0, -4), a_4 = (0, 1, 0, -2)$. There is a single firm whose production set is given by the convex cone of these activities.

1. First, utility function form gives $p_1, p_2 > 0$ in equilibrium. Normalize $p_1 = 1$. Household's optimal bundle given prices is immediate: $c^1 = \left(\frac{p_3}{2}, \frac{p_3}{2p_2}\right), c^2 = \left(p_4, \frac{p_4}{p_2}\right)$.

2. Second, either a_1 or a_2 must be used in equilibrium. Suppose neither of them is used. If $p_3 > 0$, agent 1 wants to sell good 3 and thus the market does not clear. If $p_3 = 0$, then using a_1, a_2 give positive profits. Similarly, either a_3 or a_4 must be used in equilibrium.

3. Third, a_1 cannot be used in equilibrium. Suppose a_1 is used, then $p_3 = \frac{1}{3}$. Since no activity can give strictly positive profits, we have: $p_2 \leq \frac{7}{30}, p_4 \geq \frac{1}{4}$. Thus $p \cdot a_4 = p_2 - 2p_4 < 0$, so a_4 will not be used. By the previous point, we know that a_3 must be used and hence $p_4 = \frac{1}{4}$. Note a_3 is the now only activity that takes good 4 as an input, so market clearing implies $\gamma_3 = \frac{1}{2}$. Thus the total supply of good 1 must be great than or equal to $\frac{1}{2}$. The total demand of good 1 is, however, $c_1^1 + c_1^2 = \frac{1}{6} + \frac{1}{4} < \frac{1}{2}$, so this market cannot clear.

4. Next, a_3 has to be used in equilibrium. Suppose a_3 is not used, and we know that a_1 is not used, then a_2 is the only activity that produces good 1; it is also the only activity that takes good 3 as an input. Market clearing implies the total output of good 1 is 0.1. Note non-positive profits give $p_3 \geq \frac{1}{3}$, together with $c_1^1 = \frac{p_3}{2}$, implies this market cannot clear.

5. Finally, a_4 must be used. So far, we have $\gamma_1 = 0, \gamma_2 = 1, \gamma_3 > 0$. These in turn give $p_3 \geq \frac{1}{3}$, $p_3 = p_2 + 0.1$, $p_4 = 0.25$, $p_4 \geq \frac{1}{2}p_2$. Note that $c_2^1 = \frac{p_3}{2p_2} > \frac{1}{2}$, $c_2^2 = \frac{p_4}{p_2} \geq \frac{1}{2}$, and hence total demand for good 2 exceeds 1. But if a_4 is not used, the total output for good 2 is 1, this market cannot clear. Then a_4 must be used, and hence $p_2 = 0.5, p_3 = 0.6$. Demands are $c^1 = (0.3, 0.6), c^2 = (0.25, 0.5)$. Market clearing condition for the two consumption goods pin down $\gamma_3 = 0.45, \gamma_4 = 0.1$. We can verify that input markets clear. We find a unique WE.

Recitation 4: Arrow-Debreu

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1 Equilibrium Concepts

We now extend Walras model to incorporate time and uncertainty. The key idea is to model the same physical commodities at different time and under different states as different goods. To simplify notations, we look at the two period model with uncertainty and it is straightforward to extend to multiple/infinite periods.

Assume at time $t = 0$ the economy is in state $s = 0$; at time $t = 1$ one state of nature out of S possible states of nature realizes. A bundle is a state-contingent consumption plan, $x = (x(0), x(1), \ldots, x(S)) \in \mathbb{R}_+^{L(S+1)}$. Accordingly, there are $L(S+1)$ prices and hence $L(S+1) - 1$ after normalization (say, normalize $\rho_1(0) = 1$); and a utility function should be $u^h : \mathbb{R}_+^{L(S+1)} \to \mathbb{R}$. Some common assumptions on utility functions:

- additive separability: $u^h(x) = v_0^h(x(0)) + \sum_{s=1}^S v_s^h(x(s))$
- expected utility: $u^h(x) = v_0^h(x(0)) + \sum_{s=1}^S \pi_s^h v^h(x(s))$ $\cdots\cdots\cdots\cdots\cdots\cdots (*)$
- time discounting: $u^h(x) = v^h(x(0)) + \beta \sum_{s=1}^S \pi_s^h v^h(x(s))$

An exchange economy is $E = \left(u^h, e^h\right)_{h\in\mathcal{H}}$, where $e^h = \left(e^h(0), e^h(1), \ldots, e^h(S)\right) \in \mathbb{R}_+^{L(S+1)}$.

Definition 1. A Debreu-Walras equilibrium for $E = \left(u^h, e^h\right)_{h\in\mathcal{H}}$ consists of Debreu prices $\rho \in \Delta^{L(S+1)-1}$ and an allocation $\left(x^h\right)_{h\in\mathcal{H}} \in \mathbb{R}_+^{HL(S+1)}$ such that

1. Households optimize: $\forall h, x^h \in \arg\max_x u^h(x)$ s.t. $\rho \cdot x \leq \rho \cdot e^h$.
2. Markets clear: $\sum_{h\in\mathcal{H}} \left(x^h - e^h\right) = 0$.

Remark 1. It is mathematically the same problem (with appropriate assumptions) as before, so welfare theorems and existence still hold.

Remark 2. The market structure is that trade takes place at period 0, before any uncertainty has been realized; at later periods, deliveries of the consumption goods take place but there is no re-trading. This is to sign a contract (with perfect enforcement) that decides deliveries in each period/state, but all payments happened at the time the contract was signed. This contract seems unrealistic, so Arrow reformulated the model by introducing financial securities.

Now assume that at each state s there are spot markets (i.e., immediate deliveries) for the L commodities with prices $p(s)$. There are $S+1$ spot markets; each could have normalization on spot prices. Normalize $p_1(s) = 1$ so commodity 1 is the numeraire. An *Arrow security* for state s costs α_s units of the numeraire at period 0 and pays 1 unit of the numeraire at state s and 0 at all other states.

Definition 2. An Arrow security markets equilibrium consists of prices p, α, an allocation $\left(x^h\right)_{h\in\mathcal{H}} \in \mathbb{R}_+^{HL(S+1)}$ and portfolios $\left(\theta^h\right)_{h\in\mathcal{H}} \in \mathbb{R}^{HS}$ such that

1. Households optimize: $\forall h$,

$$\begin{aligned} \left(x^h, \theta^h\right) \in \arg\max_{x,\theta} u^h(x) & \\ \text{s.t. } p(0)\cdot x(0) + \alpha\cdot\theta \leq p(0)\cdot e^h(0) & \\ p(s)\cdot x(s) \leq p(s)\cdot e^h(s) + \theta_s \quad s = 1,\ldots,S & \end{aligned} \tag{1}$$

2. Markets clear: $\sum_{h\in\mathcal{H}}\left(x^h - e^h\right) = 0$, $\sum_{h\in\mathcal{H}}\theta^h = 0$.

Remark 3. Prices of Arrow securities must be strictly positive to preclude arbitrage; otherwise, households will demand $\theta_s^h \to +\infty$ if $\alpha_s \leq 0$. The demand of Arrow securities could be either positive or negative: positive means buying and negative selling. Arrow securities do not come from nowhere: any Arrow security you buy must come from someone else who is selling it. Markets for Arrow securities clear in equilibrium. But since budget constraints hold with equality in equilibrium, all commodities market clearing automatically implies Arrow securities market clearing.

An important result is that the set of equilibria is the same with both market structures.

Theorem 1. *Prices and allocations $\left(p, \alpha, \left(x^h, \theta^h\right)_{h\in\mathcal{H}}\right)$ constitute an Arrow security markets equilibrium if and only if there exist Debreu prices $\rho \in \Delta_+^{L(S+1)-1}$ such that $\left(\rho, \left(x^h\right)_{h\in\mathcal{H}}\right)$ constitute a Walrasian equilibrium.*

Remark 4. The key is to find out the relation between prices under both markets structures. For $s = 1,\ldots,S$, multiply the budget constraint (1) in the spot market at each state s by α_s, then add them up.

$$p(0)\cdot x(0) + \alpha\cdot\theta + \sum_{s=1}^{S}\alpha_s\left(p(s)\cdot x(s)\right) \leq p(0)\cdot e^h(0) + \sum_{s=1}^{S}\alpha_s\left(p(s)\cdot e^h(s)\right) + \sum_{s=1}^{S}\alpha_s\theta_s$$

Note that the parts from Arrow security portfolios in both sides cancel out. Comparing it with $\sum_{s=0}^{S} \rho(s) \cdot x(s) \leq \sum_{s=0}^{S} \rho(s) \cdot e^h(s)$ suggests

$$p(0) = \rho(0)\,; \alpha_s p(s) = \rho(s)\,, \forall s = 1, \ldots, S. \tag{2}$$

Remark 5. Intuition for the last equation. In Arrow security markets, one can use α_s units of good 1 (numeraire) at period 0 to buy an Arrow security of state s and hence $\frac{1}{p_l(s)}$ units of good l under state s. In a Debreu-Walras market, α_s units of good 1 (we normalize $\rho_1(0) = 1$) at period 0 could afford $\frac{\alpha_s}{\rho_l(s)}$ units of good l under state s. If the two market structures are equivalent, one would expect the same trade, so

$$\frac{\alpha_s}{\rho_l(s)} = \frac{1}{p_l(s)} \tag{3}$$

In particular, take $l = 1$ and since $p_1(s) = 1$, we have

$$\alpha_s = \rho_1(s) \tag{4}$$

Remark 6. We can obtain the Arrow security market equilibrium directly from Debreu-Walras equilibrium $\left(\rho, \left(x^h\right)_{h \in \mathcal{H}}\right)$. First of all, the same allocations could be ASME by the equivalence result. Next, Arrow security prices can be obtained by equation (4). Lastly, $p_l(s)$ can be calculated from α_s and $\rho_l(s)$ using equation (3).

2 Implications

2.1 Risk Sharing

Consider the case of separable and expected utility $(*)$ and only one physical commodity. FOCs for the Pareto problem give

$$\frac{\pi_s^h v^{h\prime}\left(x^h(s)\right)}{\pi_{s'}^h v^{h\prime}\left(x^h(s')\right)} = \frac{\pi_s^{\hat{h}} v^{\hat{h}\prime}\left(x^{\hat{h}}(s)\right)}{\pi_{s'}^{\hat{h}} v^{\hat{h}\prime}\left(x^{\hat{h}}(s')\right)}\ \forall h, \hat{h}, \forall s, s' \tag{5}$$

When FWT holds, this is also true for a competitive equilibrium allocation. Assume Bernoulli utility functions are strictly increasing and strictly concave. Denote the aggregate endowment understate s by $e(s) = \sum_h e^h(s)$.

Theorem 2. *Suppose households agree on subjective probabilities: $\pi_s^h = \pi_s^{\hat{h}}, \forall h, \hat{h}, \forall s$. If $e(s) > e(s')$, then $x^h(s) > x^h(s'), \forall h$.*

Proof. If households agree on subjective probabilities, then equation (5) becomes

$$\frac{v^{h\prime}\left(x^{h}\left(s\right)\right)}{v^{h\prime}\left(x^{h}\left(s'\right)\right)}=\frac{v^{\hat{h}\prime}\left(x^{\hat{h}}\left(s\right)\right)}{v^{\hat{h}\prime}\left(x^{\hat{h}}\left(s'\right)\right)}\ \forall h,\hat{h},\forall s,s' \tag{6}$$

Now let's prove the result by contradiction. Suppose $\exists\hat{h}$ such that $x^{\hat{h}}\left(s\right)\leq x^{\hat{h}}\left(s'\right)$. Since $v^{\hat{h}}$ is strictly increasing and strictly concave, we have $v^{\hat{h}\prime}\left(x^{\hat{h}}\left(s\right)\right)/v^{\hat{h}\prime}\left(x^{\hat{h}}\left(s'\right)\right)\geq 1$. By equation (6), we also have $v^{h\prime}\left(x^{h}\left(s\right)\right)/v^{h\prime}\left(x^{h}\left(s'\right)\right)\geq 1,\forall h$. This, again by strict monotonicity and strict concavity of v^{h}, means $x^{h}\left(s\right)\leq x^{h}\left(s'\right),\forall h$, and hence $\sum_{h}x^{h}\left(s\right)\leq\sum_{h}x^{h}\left(s'\right)$. But this, together with $e\left(s\right)=\sum_{h}x^{h}\left(s\right),\forall s$, implies $e\left(s\right)\leq e\left(s'\right)$. This is a contradiction to the assumption that $e\left(s\right)>e\left(s'\right)$. □

Remark 7. This is saying that households will consume more in states with more aggregate endowments, instead of states with more individual endowments (hence "risk sharing"). This result does not rely on functional forms of utility, either: it holds even if household have different utility functions (as long as they are strictly increasing and strictly concave). It is different when one is trying to compute Walrasian equilibrium: both individual endowments and utility functions matter then.

Similarly, if there is no aggregate uncertainty, i.e., $e\left(s\right)=e\left(s'\right),\forall s,s'$, then there is no individual uncertainty: it must be that $x^{h}\left(s\right)=x^{h}\left(s'\right),\forall s,s',\forall h$. This is the most obvious way to see "risk sharing". (More generally, it can be thought of as risk sharing when the ratio of marginal utilities between two households is constant across time and states.)

This result does rely on that households agree on subjective probabilities. Suppose there are 2 households and there is no aggregate uncertainty, but they disagree on subjective probabilities. For all states s and s' such that $\frac{\pi_{s}^{1}}{\pi_{s}^{2}}>\frac{\pi_{s'}^{1}}{\pi_{s'}^{2}}$, we have $x^{1}\left(s\right)>x^{1}\left(s'\right)$. This is because

$$\frac{\pi_{s}^{1}v^{1\prime}\left(x^{1}\left(s\right)\right)}{\pi_{s'}^{1}v^{1\prime}\left(x^{1}\left(s'\right)\right)}=\frac{\pi_{s}^{2}v^{2\prime}\left(x^{2}\left(s\right)\right)}{\pi_{s'}^{2}v^{2\prime}\left(x^{2}\left(s'\right)\right)}\Longrightarrow\frac{v^{1\prime}\left(x^{1}\left(s\right)\right)}{v^{1\prime}\left(x^{1}\left(s'\right)\right)}<\frac{v^{2\prime}\left(e\left(s\right)-x^{1}\left(s\right)\right)}{v^{2\prime}\left(e\left(s'\right)-x^{1}\left(s'\right)\right)} \tag{7}$$

Suppose $x^{1}\left(s\right)\leq x^{1}\left(s'\right)$. Then $v^{1\prime}\left(x^{1}\left(s\right)\right)/v^{1\prime}\left(x^{1}\left(s'\right)\right)\geq 1$. But $e\left(s\right)-x^{1}\left(s\right)\geq e\left(s'\right)-x^{1}\left(s'\right)$ implies $v^{2\prime}\left(e\left(s\right)-x^{1}\left(s\right)\right)/v^{2\prime}\left(e\left(s'\right)-x^{1}\left(s'\right)\right)\leq 1$. This is a contradiction to inequality (7). Therefore, $x^{1}\left(s\right)>x^{1}\left(s'\right)$ and $x^{2}\left(s\right)<x^{2}\left(s'\right)$: household consumes more in states he believes relatively (compared to the other's belief) more likely to happen.

2.2 Asset Pricing

Write the Lagrangian for household problem in Arrow security markets:

$$L = u^h(x) + \lambda_0 \big(p(0) \cdot e^h(0) - p(0) \cdot x(0) - \alpha \cdot \theta\big) + \sum_{s=1}^{S} \lambda_s \big(p(s) \cdot e^h(s) + \theta_s - p(s) \cdot x(s)\big)$$

Forget about non-negativity constraints, and take first order conditions wrt $x_l(s), \forall l = 1, \dots, L, \forall s = 0, 1, \dots, S$:

$$\frac{\partial}{\partial x_l(s)} u^h(x) = \lambda_s p_l(s)$$

First order conditions wrt $\theta_s, \forall s = 1, \dots, S$

$$\lambda_0 \alpha_s = \lambda_s$$

Therefore, $\forall l = 1, \dots, L, \forall s = 1, \dots, S$

$$\frac{u^h_{l0}(x)}{p_l(0)} \alpha_s = \frac{u^h_{ls}(x)}{p_l(s)}$$

where $u^h_{ls}(x) := \frac{\partial}{\partial x_l(s)} u^h(x)$. This is the fundamental equation for asset pricing. (It is "fundamental" because once we have prices of Arrow securities, we can price any other financial security by no arbitrage, since any given financial security could be replicated by Arrow securities.) The intuition is nice: the LHS represents the marginal cost (in terms of utility) of buying one more unit of Arrow security s, whereas the RHS stands for the marginal benefit (in terms of utility) of buying one more unit of Arrow security s. A simple corollary is that MRS are equalized across households, as they face the same prices.

Note that we have endogenous variables in both sides of the equation, so it is not an explicit solution but an equilibrium condition. To obtain equilibrium prices, one needs to plug in the equilibrium allocation. This equation can be further simplified, if we recall that prices are normalized so that $p_1(s) = 1, \forall s = 0, 1, \dots, S$:

$$\alpha_s = \frac{u^h_{1s}(x)}{u^h_{10}(x)} = MRS^h_{1s,10}$$

Econ 701B Fall 2018 **University of Pennsylvania**

Recitation 5: General Equilibrium with Incomplete Asset Markets

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1 No-Arbitrage

Now we look at an economy with J financial assets. A financial asset j is characterized by a vector of dividends/payoffs denoted $d^j \in \mathbb{R}^S$: it is a contract that specifies for each state in the next period, $s = 1, \ldots, S$, the seller of the contract has to transfer d^j_s (which could be either positive or negative) units of numéraire commodities to the buyer. Arrow securities are a particular example of a financial asset.

Define the "pay-off matrix" A to have elements $a_{sj} = d^j_s$. Asset prices are denoted $q \in \mathbb{R}^J$. A portfolio is a vector $\theta \in \mathbb{R}^J$ that specifies holdings of each asset. A key idea in asset pricing is no-arbitrage: there is no free lunch. It is impossible to get a profit for sure at no cost.

Definition 1. An *arbitrage* opportunity is a portfolio $\theta \in \mathbb{R}^J$ such that $q \cdot \theta \leq 0$ and $A\theta > 0$ or such that $q \cdot \theta < 0$ and $A\theta \geq 0$. A security price system is *arbitrage free* if there does not exist such θ.

Remark 1. Assume agents have strictly increasing utility functions. We have seen before that prices must be strictly positive for households optimization to have a well-defined solution. Similarly, asset prices must preclude arbitrage to ensure that the maximization problem here has a well-defined solution. That is, no-arbitrage is a *necessary* condition for general equilibrium. It provides characterizations of asset prices without information on preferences.

Theorem 1. *(Fundamental Theorem of Asset Pricing) A security price system $q \in \mathbb{R}^J$ precludes arbitrage if and only if there exists a state price vector $\alpha \in \mathbb{R}^S_{++}$ such that $q^\top = \alpha^\top A$.*

Remark 2. It states that absence of arbitrage is equivalent to the existence of strictly positive state prices and both directions are important. One implication is that if there is no arbitrage, replicating portfolios should be priced exactly the same since they have the same payoff vector. And because of this, we assume throughout that A has full column rank J (and $J \leq S$). This provides the basis for many results in asset pricing (and hence "fundamental"): you can price any financial securities and derivatives once you obtain the payoff vector and state prices. For example, the payoff vector of a risk-free bond is $(1, 1, \ldots, 1)$ and the price of the bond that precludes arbitrage should be $\sum_{s=1}^S \alpha_s$.

Remark 3. It is not a coincidence that we use the same notation α for both state prices and Arrow security prices. When there are $J = S$ securities (whose price vector is q and payoff matrix is A), we call it a complete asset market, and GEI (defined below) is equivalent to a Arrow-Debreu equilibrium. By the Fundamental Theorem of Asset Pricing, no arbitrage means there exists a state price vector $\alpha \in \mathbb{R}^S_{++}$ such that $q^\top = \alpha^\top A$. In fact, since A is a square matrix and has full rank when $J = S$, α is unique and $\alpha^\top = q^\top A^{-1}$. The payoff matrix for Arrow securities is I, so the Arrow security prices are $\alpha^\top I$. This is just a reformulation of Arrow's model. The very point of a financial market being complete is that we can replicate the full set of Arrow securities using those financial securities and hence equivalent to Arrow-Debreu.

Remark 4. There can be many prices consistent with no-arbitrage. Whether a price system is consistent with no-arbitrage does not depend on the probability distribution of states, which does matter for equilibrium prices. Even so, two states sharing the same state price in equilibrium does not mean that they have the same probability.

Remark 5. The proof uses the separating hyperplane theorem (see slides). To illustrate why it is important to have strictly positive state prices, here is a simple "proof" which considers the case of a complete market ($J = S$). Since A is full rank, there exists a (unique) vector α such that $q^\top = \alpha^\top A$. Then we prove no-arbitrage is equivalent to $\alpha \gg 0$.

Suppose $\alpha_i \leq 0$ for some i. Consider a portfolio θ such that

$$A\theta = \Big(0, \ldots, 0, \underbrace{1}_{i\text{th}}, 0, \ldots, 0\Big)^\top > 0$$

(again, its existence is guaranteed by full rank). Then $q^\top \theta = \alpha^\top A\theta = \alpha_i \leq 0$, which violates arbitrage free.

Consider the other direction and suppose q is subjective to arbitrage, that is, we can find a portfolio θ such that $q \cdot \theta < 0$ and $A\theta \geq 0$ (or $q \cdot \theta \leq 0$ and $A\theta > 0$), but $q^\top \theta = \alpha^\top A\theta$ implies that there must be some $\alpha_i \leq 0$.

Exercise 1. Suppose there are 3 states. At date 0, agents can trade in two primary securities with prices $q_1 = 0.1, q_2 = 1.1$ and second-period payoff vectors $d^1 = (1, 0, 0)', d^2 = (1, 2, 3)'$.

1. Suppose there is a call option on security 2 with strike price of 1. What possible prices are consistent with no arbitrage?

2. Suppose $q_3 = 1$. Find an arbitrage opportunity.

3. Assume these security prices $q = (0.1, 1.1, 0.6)$. What would be the market price of a put option on asset 2 with a strike price of 3? What would be the risk-free interest rate on a loan taken at date 0?

Solution.

1. The payoff vector of the call option is $d^3 = (0, 1, 2)$. By the Fundamental Theorem of Asset Pricing, if prices are consistent with no arbitrage, then there exists state prices $\alpha^\top = (\alpha_1, \alpha_2, \alpha_3) \gg 0$ such that

$$(0.1, 1.1, q_3) = (\alpha_1, \alpha_2, \alpha_3) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 3 & 2 \end{bmatrix} \implies \begin{cases} \alpha_1 = 0.1 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 = 1.1 \\ \alpha_2 + 2\alpha_3 = q_3 \end{cases}$$

   So $\alpha_1 = 0.1, 2\alpha_2 + 3\alpha_3 = 1$ and $\alpha_2 \in \left(0, \frac{1}{2}\right), \alpha_3 \in \left(0, \frac{1}{3}\right)$. Therefore, $q_3 = \frac{1}{2}(1 - 3\alpha_3) + 2\alpha_3 = \frac{1}{2} + \frac{1}{2}\alpha_3 \in \left(\frac{1}{2}, \frac{2}{3}\right)$.

2. From (1) we know $q_3 = 1$ does not preclude arbitrage. Here is an arbitrage strategy: buy one unit of security 2, and sell one unit of security 1 and 3. At date 0, the cost of this portfolio is $1.1 - 0.1 - 1 = 0$. At date 1, the payoff vector is $(0, 1, 1) > 0$.

3. From (1) we know this price vector is consistent with no arbitrage. By the Fundamental Theorem of Asset Pricing, we can solve for the state prices:

$$(0.1, 1.1, 0.6) = (\alpha_1, \alpha_2, \alpha_3) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 3 & 2 \end{bmatrix} \implies \begin{cases} \alpha_1 = 0.1 \\ \alpha_2 = 0.2 \\ \alpha_3 = 0.2 \end{cases}$$

   A put option on asset 2 with a strike price of 3 has a payoff vector $(2, 1, 0)'$. The arbitrage-free price should be $(0.1, 0.2, 0.2) \cdot (2, 1, 0) = 0.4$.

   A risk-free bond has a payoff vector $(1, 1, 1)'$, the arbitrage-free price is $(0.1, 0.2, 0.2) \cdot (1, 1, 1) = 0.5$. Hence the interest rate is $1/0.5 - 1 = 100\%$.

Exercise 2. For this question, it is extremely helpful to use Linear Programming type graphs. I will Illustrate it on the blackboard. The idea is that the known prices give us linear constraints about some state prices, and the unknown price is a linear function of those state prices. Draw the feasible area in a Cartesian coordinate system, and find out the range (i.e. max and min) of the linear objective function. Even if you cannot immediately identify the maximum and minimum, the following fact in linear programming guarantees that you can simply try out the corners of the feasible area:

Fact. *Every linear program has an extreme point that is an optimal solution.*

Corollary. *An algorithm to solve a linear program only needs to consider extreme points.*

1. Suppose there are 3 states and 2 assets. The assets pay

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \\ 3 & 0 \end{pmatrix}$$

   If the price of asset 1 is 1, what are the prices of asset 2 that are consistent with no arbitrage?

2. Suppose there are 4 states and 3 assets. The assets pay

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 3 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

   If the prices of assets 1 and 2 are both 1, what are the possible prices for asset 3 that are consistent with no arbitrage?

Example 1. A useful trick. If (q_1, q_2, q_3) and the first payoff matrix precludes arbitrage,

$$\begin{pmatrix} \underline{q_1} & \underline{q_2} & \underline{q_3} \\ 1 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} \underline{q_1+q_3} & \underline{q_2} & \underline{q_3} \\ 2 & 1 & 1 \\ 5 & 0 & 3 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \\ 4 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} \underline{q_1+q_3} & \underline{q_2} & \underline{q_3} \\ 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & \frac{3}{5} \\ 1 & 3 & 0 \\ 1 & 0 & 1 \\ 1 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \longrightarrow \begin{pmatrix} \underline{1} & \underline{\frac{q_2}{q_1+q_3}} & \underline{\frac{q_3}{q_1+q_3}} \\ 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & \frac{3}{5} \\ 1 & 3 & 0 \\ 1 & 0 & 1 \\ 1 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

then $\left(1, \frac{q_2}{q_1+q_3}, \frac{q_3}{q_1+q_3}\right)$ and the last payoff matrix also does. This implies that there exists positive state prices such that $\sum_{s=1}^{5} \alpha_s = 1$ and

$$\left(\frac{q_2}{q_1+q_3}, \frac{q_3}{q_1+q_3}\right) = \alpha_1 \left(\frac{1}{2}, \frac{1}{2}\right) + \alpha_2 \left(0, \frac{3}{5}\right) + \alpha_3 (3, 0) + \alpha_4 (0, 1) + \alpha_5 \left(\frac{1}{4}, \frac{1}{2}\right)$$

which means $\left(\frac{q_2}{q_1+q_3}, \frac{q_3}{q_1+q_3}\right)$ is in the interior of the convex cone spanned by the five points.

2 Incomplete Asset Markets

If $J < S$, the asset markets are incomplete. It is widely believed that asset markets are incomplete (moral hazard, transaction costs, or some financial instruments simply have not been invented, etc).

Definition 2. A General equilibrium with (potentially) incomplete asset markets (GEI) is a collection of portfolio-holdings $\left(\theta^h\right) \in \mathbb{R}^{HJ}$, households consumption $\left(x^h\right)_{h\in\mathcal{H}} \in \mathbb{R}_+^{HL(S+1)}$, spot prices $(p(s))_{s=0}^S$ and asset prices $q \in \mathbb{R}^J$ such that

1. Household optimization: $\forall h \in \mathcal{H}$,

$$\begin{aligned}\left(x^h, \theta^h\right) \in \arg &\max_{\theta\in\mathbb{R}^J, x\in\mathbb{R}_+^{L(S+1)}} u^h(x) \\ \text{s.t. } &p(0)\cdot x(0) \leq p(0)\cdot e^h(0) - q\cdot\theta \\ &p(s)\cdot x^h(s) \leq p(s)\cdot e^h(s) + \sum_{j\in\mathcal{J}} d_s^j\theta^j \quad s = 1,\ldots,S\end{aligned}$$

2. Market clearing: $\sum_{h\in\mathcal{H}}\theta^h = 0$ and $\sum_{h\in\mathcal{H}}\left(x^h - e^h\right) = 0$.

For existence and welfare theorems of GEI, 2011 Final Q3 is a good example for reference. Existence theorem still holds for incomplete markets economy under the standard assumptions of utility functions and initial endowments: under weak assumptions on the asset payoff matrices, assuming strictly increasing and strictly concave utility and interior endowments, standard methods can be used to prove that an equilibrium in our economy always exists. First welfare theorem no longer holds if markets are not complete. In fact, GEI is generically Pareto inefficient. Second welfare theorem still holds.

Example 2. An example for the failure of FWT. Consider an economy with 2 households, 2 physical goods, and 2 periods with 2 states in the second period. Suppose $U^h(x) = u^h(x_0) + \pi_1 u^h(x_1) + \pi_2 u^h(x_2)$, where $u^h(x_s) = \ln x_{s1} + \ln x_{s2}$. Endowments are $e_0^1 = e_1^1 = e_2^1 = e_0^2 = e_1^2 = (1,1)$, $e_2^2 = (0,0)$. Spot markets exist, but there is only Arrow security for state 1. Autarky is an equilibrium with prices $p_{01} = p_{02} = p_{11} = p_{12} = p_{21} = p_{22} = 1$ and $\alpha_1 = \pi_1$. But this is not Pareto efficient: there exists small $0 < \epsilon, \delta < 1$ such that

$$\begin{aligned}U^1((1,1,1+\epsilon,1+\epsilon,1-\delta,1-\delta)) &\geq 0 = U^1\left(e^1\right) \\ U^2((1,1,1-\epsilon,1-\epsilon,\delta,\delta)) &> -\infty = U^2\left(e^2\right)\end{aligned}$$

Recitation 6: Core

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So far we have assumed price taking, where agents will not realize that his behavior will affect prices. Now we introduce a solution concept "*core*", reflecting a notion of stability, to take into account interactions: no subgroup of agents can use their own endowments and find an allocation in which each one of them is better off (with one strictly better off). Note that it is different from the main focus of 703, which takes a non-cooperative approach and concerns unilateral deviation; here it is a cooperative approach to think about "*blocking coalitions*".

Definition 1. Given an exchange economy E, a feasible allocation $\left(c^h\right)_{h\in\mathcal{H}}$ is in the *core* if there does not exist a *blocking coalition*, i.e., there does not exist a subset of consumers $\mathcal{S}\subset\mathcal{H}$ with consumptions $\left(\tilde{c}^h\right)_{h\in\mathcal{S}}$ such that $\sum_{h\in\mathcal{S}}\tilde{c}^h\leq\sum_{h\in\mathcal{S}}e^h$ and such that $u^h\left(\tilde{c}^h\right)\geq u^h\left(c^h\right)$ for all $h\in\mathcal{S}$ where the inequality holds strict for at least one.

Theorem 1. *All core allocations are Pareto-efficient.*

Proof. Contrapositive: Pareto efficiency is saying that $\mathcal{H}$ cannot form a blocking coalition. So Pareto inefficiency implies that $\mathcal{H}$ is a blocking coalition. □

Now we provide two theorems that are parallel to FWT and SWT.

Theorem 2. *Assume (A2) or (LNS). A Walrasian equilibrium allocation is in the core.*

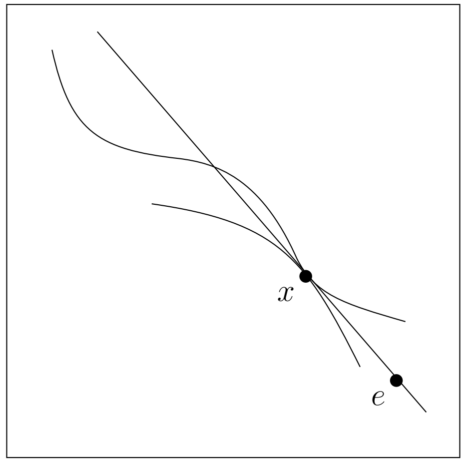
Proof. The proof strategy is essentially parallel to FWT and proceeds *ad absurdum.* Suppose $x^*=(x_1^*,\ldots,x_H^*)$ is a Walrasian allocation with corresponding price p^*, but not in the core. That is, $\exists S\subset\mathcal{H}$ and $\{y_i\}_{i\in S}$ with

$$y_i\succsim_i x_i^*,\forall i\in S\quad\text{and}\quad y_j\succ_j x_j^*,\exists j\in S$$

Since x^* is a Walrasian allocation, $p^*\cdot y_i^*\geq p^*w_i,\forall i\in S$ and $p^*\cdot y_j^*>p^*w_j,\exists j\in S$. But this contradicts with $\sum_{i\in S}\left(y_i^*-w_i^*\right)\leq 0$. □

Remark 1. This guarantees that the core is nonempty for any pure exchange economy for which we know there exists a Walrasian equilibrium. If agents' preferences are not convex, there may not be a Walrasian equilibrium, but there may still exist an allocation in the core (see Figure (1) for an example). In fact, the result of core being nonempty is quite general.

Figure 1: Example for Nonexistence of WE But Core Nonempty



One may try to prove that for a two-agent two-good exchange economy where agents' utility functions are continuous and strictly increasing (but not necessarily concave), the core is nonempty.

Proof. Consider a maximization problem

$$\max_{x_1, x_2 \in \mathbb{R}_+^2} u^1(x_1, x_2) \text{ s.t. } x_1 \leq e_1, x_2 \leq e_2, \text{ and } u^2(e_1 - x_1, e_2 - x_2) \geq u^2(e^2)$$

The constraint set, denoted K, is nonempty, bounded and closed (by continuity of u). So the above maximization problem has a solution, denoted (c_1^1, c_2^1). Define $c_l^2 = e_l - c_l^1$ and we will show that allocation (c^1, c^2) is in the core hence nonempty.

First, we prove that there is no singleton blocking coalition. Since $e^1 \in K$, by definition $u^1(c^1) \geq u^1(e^1)$. And trivially $u^2(c^2) \geq u^2(e^2)$ by the constraint. So each agent cannot form a blocking coalition by herself.

Next, we prove that two agents together cannot form a blocking coalition either, and proceed by contradiction. Suppose there is a feasible allocation $(\tilde{c}^1, \tilde{c}^2)$ such that $u^i(\tilde{c}^i) \geq u^i(c^i)$ for each i, with the inequality being strict for at least one agent.

If $u^1(\tilde{c}^1) > u^1(c^1)$, then $\tilde{c}^1 \notin K$ by definition of c^1. Since $\tilde{c}^1$ is feasible and satisfies the first two constraints, it must violate the last one, and $u^2(\tilde{c}^2) < u^2(e^2)$. But this is a contradiction. So $u^2(\tilde{c}^2) > u^2(c^2)$ and $u^1(\tilde{c}^1) = u^1(c^1)$.

Since u^2 is strictly increasing, at least one of $\tilde{c}_1^2$ and $\tilde{c}_2^2$ is strictly positive. WLOG, assume $\tilde{c}_1^2 > 0$. Since u^2 is continuous, $u^2(\tilde{c}_1^2 - \varepsilon, \tilde{c}_2^2) > u^2(c^2)$ for sufficiently small $\varepsilon > 0$. However, this implies that $(\tilde{c}_1^1 + \varepsilon, \tilde{c}_2^1) \in K$ while it gives agent 1 higher utility than c^1, which is a contradiction. $\square$

Remark 2. Given Theorem 1 and 2, FWT is a straightforward corollary. The assumption needed is the same as FWT: A2 or LNS. An example of an economy in which there is a Walrasian equilibrium allocation that is not in the core violates local nonsatiation. For example, a two-person two-good economy where $u^1(x_1, x_2) = 1, u^2(x_1, x_2) = x_1 x_2$ and $e^1 = e^2 = (1, 1)$. Then no trade can be a Walrasian equilibrium under $p = (1, 1)$. But this is not Parento efficient hence not in the core, as giving all the goods to agent 2 makes him better off without harming agent 1.

We would like to have some result akin to SWT, about whether core allocation can be supported as an equilibrium. To formalize this result, we need one more definition of replica:

Definition 2. Start with an exchange economy with H households $E^1 = \left(u^h, e^h\right)_{h\in\mathcal{H}}$. For each number $n = 1, 2, 3, \ldots$, we can construct an economy E^n (*replica*) with nH households where there are n identical consumers of each *type* h such that

$$u^{(h,i)} = u^{(h,j)} = u^h, \quad e^{(h,i)} = e^{(h,j)} = e^h$$

for all $i, j = 1, \ldots, n$ of the same type h.

Lemma 1. *("Equal Treatment" Property) Assume (A4'). For each $n \geq 0$, if $\left(c^{(h,i)}\right) h \in \mathcal{H}, i = 1, \ldots, n$ is in the core, then $c^{(h,i)} = c^{(h,j)}$ for all $i, j = 1, ., n$.*

Proof. Pick the least desired (under u^h) consumption which occurs for each type h, denoted $\tilde{c}(h)$. Suppose there is type $\bar{h}$ such that $c^{(\bar{h},i)} \neq c^{(\bar{h},j)}$ for some i, j. We can prove that the worst agents among each type can form a blocking coalition.

By strict concavity

$$u^h\left(\frac{1}{n}\sum_{i=1}^{n} c^{hi}\right) \geq \frac{1}{n}\sum_{i=1}^{n} u^h\left(c^{hi}\right) \geq u^h\left(\tilde{c}\left(h\right)\right), \forall h$$
$$u^{\bar{h}}\left(\frac{1}{n}\sum_{i=1}^{n} c^{\bar{h}i}\right) > \frac{1}{n}\sum_{i=1}^{n} u^{\bar{h}}\left(c^{\bar{h}i}\right) \geq u^{\bar{h}}\left(\tilde{c}\left(\bar{h}\right)\right)$$

By feasibility, $\sum_{h=1}^{H}\left(\frac{1}{n}\sum_{i=1}^{n} c^{hi} - e^h\right) = 0$. The set consisting of one consumer of each type, each of whom is the worst off would block the allocation - a contradiction. □

This lemma, together with the fact that equilibrium allocations are in the core, implies that equilibrium consumption is equal across individuals of the same type. Furthermore, the lemma implies that the equilibrium allocations and prices are the same for all n.

Theorem 3. *(Core Convergence Theorem) (Under standard assumptions) If $x^* = (x_1^*, \ldots, x_H^*)$ is the core for every replica, x^* is Walrasian.*

Remark 3. The proof idea is that for any allocation x that is not Walrasian, we can find a sufficiently large replica where x is not in its core. This theorem provides a good justification for the competitive paradigm: as the number of households is getting larger, any allocation that is always in the core, can also be obtained as a Walrasian equilibrium.

Summary.

- Definition: no blocking coalition.
- W.E. is in the core: parellel to FWT proof.
- Equal Treatment lemma: same type gets the same allocation in the core (proof idea: pick the worst agents among each type to form a blocking coalition).
- Core Convergence Theorem: if an allocation is in the core for every replica, it is W.E.

Recitation 7: Public Good

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So far we have been talking about private goods. Now we turn to public goods, which are different from private goods in two properties: first, public goods are non-excludable: it cannot be denied a given agent; second, public goods are non-rival: consumption by one agent does not reduce the possibility of consumption by other agents.

1 Optimal Provision

The optimal provision of public good solves

$$\begin{aligned} \max_{\{x,y,z\}\geq 0} & u_1\left(x_1, y\right) \\ & u_i\left(x_i, y\right) \geq \underline{u}_i \quad \text{for } i = 2, 3, \ldots, n \\ & E - \sum_{h=1}^{n} x^h - z \geq 0 \\ & f\left(z\right) - y \geq 0 \end{aligned}$$

From Interior Kuhn-Tucker conditions we can obtain

$$\sum_{h=1}^{n} \frac{\partial u^h\left(x^h, y\right)/\partial y}{\partial u^h\left(x^h, y\right)/\partial x^h} = \frac{1}{f'(z)}$$

which can be written as $\sum_h MRS^h = MRT$ (*Samuelson condition*).

Remark 1. MRS^h, measuring the quantity of private goods household h would be willing to give up for an additional unit of public good, is a notion of marginal benefit for household h from consuming the public good in terms of the private good. $\sum_h MRS^h$ is hence the social marginal benefit. MRT, measuring the quantity of private good that is actually required to produce the additional unit of public good, is a notion of marginal cost in producing the public good in terms of the private good. Using Econ 1 language, Samuelson condition is saying that social optimum means social marginal benefit = social marginal cost.

2 Private Provision

Suppose household can choose to contribute part of their endowments to public good production. This translates to household h solving

$$\max_{0 \le z^h \le e^h} u^h \left(e^h - z^h, f\left(z^h + \sum_{j \ne h} z^j \right) \right)$$

which leads to $\frac{\partial u^h / \partial y}{\partial u^h / \partial x^h} = \frac{1}{f'(z)}$ (or, $MRS^h = MRT$) for interior solutions.

Alternatively, we could formulate the above idea in competitive markets for private and public goods. Firm solves $\max_{z \ge 0} pf(z) - z$, which gives $p = \frac{1}{f'(z)}$. Household h solves, after obvious simplifications,

$$\max_{0 \le y^h \le e^h / p} u^h \left(e^h - py^h, y^h + \sum_{j \ne h} y^j \right)$$

which gives $\frac{\partial u^h / \partial y}{\partial u^h / \partial x^h} = p$ for interior solutions[1].

But either formulation means $\sum_h MRS^h > MRT$, which violates Samuelson condition. We have under-provision of the public good (MRS_{yx} being too high translates to y being too little, x being to much). This inefficiency comes from the positive externality that agents do not consider the benefit to others when making decisions.

3 Lindahl Equilibrium

The idea of Lindahl equilibrium is to charge each agent a personalized price (which adds up to a total price to firms), and agents agree on the level of public good.

Definition 1. A Lindahl equilibrium is $\left(\left(p^{h*}\right)_{h \in \mathcal{H}}, \left(x^{h*}\right)_{h \in \mathcal{H}}, y^* \right)$ such that

- Firm optimization

$$y^* = \arg\max_{y \ge 0} \left(\sum_h p^{h*} \right) y - f^{-1}(y)$$

[1] A more formal argument should be the following. Assume u^h satisfies Inada condition regarding x, then x cannot be a cornor solution, and F.O.C. gives $\frac{\partial u^h / \partial y}{\partial u^h / \partial x^h} \le p$ for all h. Assume we also have Inada condition regarding y, then y cannot be 0 under maximization (note that it is not saying that y^h cannot be 0), so there must be some h buying the public good, and hence $\frac{\partial u^h / \partial y}{\partial u^h / \partial x^h} = p$.

- Households optimization

$$
\begin{aligned}
\left(x^{h*}, y^*\right) &= \arg\max_{x^h, y} u^h\left(x^h, y\right) \\
\text{s.t. } & e^h + s^h\left(\sum_h p^{h*}y^* - f^{-1}\left(y^*\right)\right) - x^h - p^{h*}y \geq 0
\end{aligned}
$$

- Market clearing

$$
\sum_h x^{h*} + f^{-1}\left(y^*\right) \leq \sum_h e^h
$$

Firm FOC gives $\sum_h p^{h*} = \frac{1}{f'(f^{-1}(y^*))}$. Households FOC gives $\frac{\partial u^h\left(x^{h*}, y^*\right)/\partial y}{\partial u^h\left(x^{h*}, y^*\right)/\partial x^h} = p^{h*}$. They together imply

$$
\sum_h \frac{\partial u^h\left(x^{h*}, y^*\right)/\partial y}{\partial u^h\left(x^{h*}, y^*\right)/\partial x^h} = \sum_h p^{h*} = \frac{1}{f'\left(f^{-1}\left(y^*\right)\right)}
$$

so the Samuelson condition is verified.

Proposition 1. *A Lindahl equilibrium allocation is Pareto optimal.*

Remark 2. It is in every consumer's interest to understate his desire for the public good. Truth telling is not an equilibrium.

4 A Big Picture for the Course

In 701A, we formalize individual decision problem (e.g. for consumers, for firms, and under uncertainty): how to pick the "best" alternative under a given constraint. Such a problem naturally requires a well-defined notion for what we mean by the "best", i.e., a way to compare various feasible alternatives. This is where "preferences" come in.

701B brings together different agents, where all households and firms are optimizing, and in addition, they achieve a notion of consistency, i.e., market clearing. This is the idea of Walrasian equilibrium, which lies at the core of 701B. FWT provides justification for WE being "good" in the sense that every WE is PO. [Here, the notion of being "good" is a little bit vague – now we are comparing different allocations, and need to define a collective preference. Going further takes you to the problem of aggregation of preferences and social choice theory.] SWT saves WE more: name any interior PO you like, and it could be supported as a WE. Then we prove existence.

Though we start with an exchange economy, we could add production into the model. Linear technology is an interesting example. We could also extend the model to incorporate dynamics and/or uncertainty. We have Debreu's formulation and Arrow's formulation, and it turns out that they are equivalent. We can see a risk sharing property from this model. We can derive an equilibrium condition characterizing Arrow security prices, and thus can price any financial security by no arbitrage. No arbitrage is a necessary condition for equilibrium. (We have seen some other "no free lunch" ideas parallel to this one: prices cannot be negative in equilibrium; a linear activity cannot generate positive profits in equilibrium.) We further relax the assumption to consider general equilibrium under incomplete asset markets, where FWT generally fails. If the asset market is complete, we go back to Arrow-Debreu model.

Next, we look at a notion of stability. An allocation is in the core if there is no blocking coalition. PO is a necessary condition for being in the core. Equal treatment is another implication if assuming strict concavity. We have two results akin to FWT and SWT: first, WE is in the core; second, if an allocation is in the core for every replica, it is Walrasian.

So far we have been looking at commodities markets, where prices are such that you can get any good you desire as long as you pay the price; and prices clear the market. In matching markets, however, prices do not work in this way. We briefly introduce matching.

Public goods give an interesting case where FWT fails. Private provision under a standard market mechanism leads to inefficient (and too low) provision of the public goods. Lindahl equilibrium is a trial to achieve optimum in a decentralized institution. [Thinking further of the incentive compatibility takes you to the problem of implementation and mechanism design theory.]

Econ 701B Fall 2018

Professor Andrew Postlewaite **University of Pennsylvania**

Suggested Solutions to Final 2018

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Question 1. Consider a two-period economy with two agents and a single consumption good. Agent h's preferences over consumption streams $\left(c_1^h, c_2^h\right)$ (c_i^h is the consumption of the good by h in period i) are represented by the separable utility function $u^h\left(c_1^h\right) + \delta u^h\left(c_2^h\right)$ where $u^h(\cdot)$ is strictly increasing, strictly concave and differentiable, $h = 1, 2$, and $u^{h\prime}(0) = \infty$. Both agents have strictly positive endowments in each period and the aggregate endowment of the good in period 1 is strictly greater than the aggregate endowment in period 2.

(a) Show $c_1^h > c_2^h$ in a competitive equilibrium for this economy for $h = 1, 2$.

Solution. Since utility is strictly increasing, prices must be positive, and hence can be normalized to $(1, p)$. Inada condition helps rule out corner solutions, so in a competitive equilibrium we have

$$\frac{u^{1\prime}\left(c_1^1\right)}{\delta u^{1\prime}\left(c_2^1\right)} = \frac{u^{2\prime}\left(c_1^2\right)}{\delta u^{2\prime}\left(c_2^2\right)} = \frac{1}{p} \tag{1}$$

which implies

$$\frac{u^{1\prime}\left(c_1^1\right)}{u^{1\prime}\left(c_2^1\right)} = \frac{u^{2\prime}\left(e_1 - c_1^1\right)}{u^{2\prime}\left(e_2 - c_2^1\right)} \tag{2}$$

by market clearing conditions. Suppose the statement is not true, and without loss of generality, suppose $c_1^1 \leq c_2^1$. Since $u^h(\cdot)$ is strictly increasing and strictly concave, this implies $\frac{u^{1\prime}\left(c_1^1\right)}{u^{1\prime}\left(c_2^1\right)} \geq 1$. Because $e_1 > e_2$, we have $e_1 - c_1^1 > e_2 - c_2^1$ and hence again by strict monotonicity and strict concavity of $u^2(\cdot)$, $\frac{u^{2\prime}\left(e_1 - c_1^1\right)}{u^{2\prime}\left(e_2 - c_2^1\right)} < 1$. But this is a contradiction.

(b) How does the answer to part a change if the good is storable, that is, endowment in period 1 can be held over to period 2?

Solution. Now there is one more ingredient in households' problem: they decide the amount of the good in period 1 to be carried over to period 2. (It is equivalent to model that everyone has a linear technology which can produce 1 unit of good in Period 2 using 1 unit of good in Period 1 as an input.) Or formally,

$$\begin{aligned}\left(c_1^h, c_2^h, s^h\right) = &\arg \max_{(c_1, c_2, s) \geq 0} u^h(c_1) + \delta u^h(c_2) \\ &\text{s.t.}\ (c_1 + s) + p(c_2 - s) \leq e_1^h + p e_2^h\end{aligned}$$

The market clearing condition are modified to $\sum_h \left(c_1^h + s^h\right) = e_1$, $\sum_h \left(c_2^h - s^h\right) = e_2$.

- If $p > 1$, no one will buy the good in Period 2 as they can always store the good from Period 1 which is cheaper, so the second market clearing condition cannot hold. (This can be thought of as the result that a linear activity cannot generate positive profit in equilibrium.)

- If $p < 1$, we will have $s^1 = s^2 = 0$. (This is essentially the result that a linear activity with negative profit will not be used in equilibrium.) Then we get the same condition as equation (2) in (a). Therefore by the same logic, $c_1^h > c_2^h, \forall h = 1, 2$. We have $p > \delta$ in equilibrium.

- If $p = 1$, then household problem F.O.C. gives $\frac{u^{h\prime}\left(c_1^h\right)}{\delta u^{h\prime}\left(c_2^h\right)} = \frac{1}{p} = 1$. Then $0 < \delta < 1$ implies $u^{h\prime}\left(c_1^h\right) < u^{h\prime}\left(c_2^h\right)$ and hence $c_1^h > c_2^h, \forall h = 1, 2$.

That is, we still have $c_1^h > c_2^h$ in a competitive equilibrium for $h = 1, 2$.

(c) How would the answer to part b change if the aggregate endowment of the good in period 1 is strictly less than the aggregate endowment in period 2?

Solution. Notice that $c_1^1 + c_1^2 = e_1 - s < e_2 + s = c_2^1 + c_2^2$, where $s \geq 0$ denotes the aggregate storage. Then from the equilibrium condition (1) we can show that it must be $c_1^h < c_2^h, \forall h = 1, 2$ by strict monotonicity and strict concavity of the utility function.

You can alternatively argue through prices. As in (b), we know that it cannot be that $p > 1$. We can also argue that $p \neq 1$, as otherwise household problem F.O.C. gives $c_1^h > c_2^h$, and hence $e_1 = \sum_h \left(c_1^h + s^h\right) > \sum_h \left(c_2^h - s^h\right) = e_2$, which is a contradiction to $e_1 < e_2$. Therefore, $p < 1$, and thus $s^1 = s^2 = 0$. We will get the same condition as equation (2) in (a), except for that now we have $e_1 < e_2$. By the same logic as above, $c_1^h < c_2^h, \forall h = 1, 2$. We have $p < \delta$ in equilibrium.

Question 2. Two farmers face the possibility that the river on which their farms lie might flood. For simplicity suppose that either of their farms might flood, but not both. The chance that either farm might flood is 1/4. Each farmer's crop will be 200 if his farm doesn't flood and 0 if it does flood. Each has a von Neumann-Morgenstern utility function with utility for the good being $u(x) = \ln x$.

(a) Compute the Arrow-Debreu equilibrium for this economy, where the farmers can trade contingent commodities before it is known whose farm might flood. What is the expected utility of each farmer?

Solution. Define three states as: (1) farmer 1's field is flooded; (2) farmer 2's field is flooded; (3) neither is flooded. Then endowments can be written as

$$e^1 = (0, 200, 200), e^2 = (200, 0, 200)$$

The aggregate endowments are $e = (200, 200, 400)$.

Note that ln utility (with positive probabilities) leads to positive prices and rules out corner solutions. In Arrow-Debreu equilibrium, we have $\forall h = 1, 2$, and $\forall s, t \in \{1, 2, 3\}$,

$$\frac{\pi_s \left(x_s^h\right)^{-1}}{\pi_t \left(x_t^h\right)^{-1}} = \frac{p_s}{p_t} \tag{3}$$

which implies $\frac{x_t^1}{x_s^1} = \frac{x_t^2}{x_s^2} = \frac{x_t^1+x_t^2}{x_s^1+x_s^2} = \frac{e_t}{e_s}$, and hence $\frac{p_s}{p_t} = \frac{\pi_s}{\pi_t}\frac{e_t}{e_s}$. Therefore, the price vector is $(1, 1, 1)$ after normalization. We can derive the equilibrium allocation (very quickly if you exploit that two farmers are equally wealthy)

$$x^1 = x^2 = (100, 100, 200)$$

and hence the expected utility of each farmer is

$$U^1 = U^2 = \frac{1}{4}\ln 100 + \frac{1}{4}\ln 100 + \frac{1}{2}\ln 200 = \ln\left(100\sqrt{2}\right)$$

(b) Suppose now that there is probability 0 that farmer 1's field will be flooded but the probability that farmer 2's field will be flooded is still 1/4. How would your answer to part (a) change?

Solution. Now the probabilities become $\pi_1 = 0, \pi_2 = 1/4, \pi_3 = 3/4$. Note that consumption under state 1 will not add to utility. If $p_1 > 0$, we will have $x_1^1 = x_1^2 = 0$ and the market for state 1 cannot clear. Thus it must be $p_1 = 0$. For the other two states, we have the same condition as equation (3) in (a), except for now $s, t \in \{2, 3\}$. Similarly, we will have $\frac{x_2^1}{x_3^1} = \frac{x_2^2}{x_3^2} = \frac{e_2}{e_3} = \frac{1}{2}$, and hence $\frac{p_2}{p_3} = \frac{\pi_2}{\pi_3}\frac{e_3}{e_2} = \frac{2}{3}$. The price vector is $(0, 2, 3)$.

The utility functional form gives a quick way to write down the Marshallian demand (i.e. the probabilities of a given state will give the shares of wealth spent on consumption under this state). Under this price vector, farmer 1's demand would be $x^1 = (x_1^1, 125, 250)$ and farmer 2's demand would be $x^2 = (x_1^2, 75, 150)$. Anything such that $x_1^1 + x_1^2 = 200, x_1^1 \geq 0, x_1^2 \geq 0$ makes it an equilibrium. Now the expected utility is

$$U^1 = \frac{1}{4}\ln 125 + \frac{3}{4}\ln 250$$

$$U^2 = \frac{1}{4}\ln 75 + \frac{3}{4}\ln 150$$

(c) Suppose now that weather forecasting becomes perfected so that it will be known whether or not there will be a flood at the time the contingent claims markets open. What will be known is whether there will be a flood, but not which farmer will be affected should there be a flood. How will this affect the ex ante utilities of the farmers?

Solution. When the weather forecasting predicts that there will not be a flood, farmers will just consume their endowments. When the weather forecasting predicts that there will be a flood, the conditional probabilities for each state are $\pi_1 = 1/2, \pi_2 = 1/2, \pi_3 = 0$. Under these conditional probabilities, we can obtain that $p = (1, 1, 0)$ and $x_1^1 = x_1^2 = 100$, $x_2^1 = x_2^2 = 100$ (repeating the previous exercise). Therefore, the ex ante utilities of the farmers are

$$U^1 = U^2 = \frac{1}{4}\ln 100 + \frac{1}{4}\ln 100 + \frac{1}{2}\ln 200 = \ln\left(100\sqrt{2}\right)$$

which is the same as in (a).

Question 3. Consider an exchange economy with two consumers and two goods. Good x is a perfectly divisible numeraire. Good y, in contrast, is *indivisible*, that is, consumers can only consume it in nonnegative integer amounts. The utility of consumer $i = 1, 2$ from consuming a bundle (x^i, y^i) of the two goods is given by $u^i(x^i, y^i) = x^i + v^i(y^i)$, where $v^i(\cdot)$ is a function on nonnegative integers. Assume that

$$v^i(2) > v^i(1) = v^i(0) = 0, \text{ and } v^i(y) = v^i(2) \text{ for } y > 2$$

(Think of good y as chopsticks where the value of only one is 0.) Assume also that

$$v^2(2) \le v^1(2) \le 10$$

The initial endowment of consumer $i = 1, 2$ is $\left(e_x^i, e_y^i\right)$. Assume the total endowment of good y is $e_y^1 + e_y^2 = 2$, and that $e_x^1 = e_x^2 = 20$.

(a) Describe the Pareto efficient allocations in this economy.

Solution. The obvious necessary conditions are: there is no waste for x, i.e., $x^1 + x^2 = 40$; and, either one agent gets both units of y, i.e.,$y^1 = 2, y^2 = 0$ or $y^1 = 0, y^2 = 2$.

- If $v^1(2) = v^2(2)$, these conditions are also sufficient: they describe all Pareto efficient allocations.

- If instead $v^1(2) > v^2(2)$, an allocation is Pareto efficient if and only if it is non-wasteful, and either agent 1 gets both units of y; or agent 2 gets both units and $x^1 < v^2(2)$. (Note that if agent 2 gets both units of y and $x^1 \geq v^2(2)$, we can transfer $v^2(2)$ units of x from agent 1 to agent 2 and let agent 1 get both units of y to achieve a Pareto improvement.)

(b) Write conditions for a Walrasian equilibrium for this economy.

Solution. First of all, prices must be strictly positive. If price of x is 0, agents will demand infinite amount of x. If price of y is 0, each agent will demand at least 2 units of y, so market demand would exceed supply. Thus we can normalize prices to be $p_x = 1$ and $p_y = p > 0$.

- If $2p > v^1(2) \geq v^2(2)$, no body will demand y so the market cannot clear.
- If $2p < v^2(2) \leq v^1(2)$, each agent will demand both units of y as long as he can afford them. Indeed, each agent's wealth is at least $20 > 10 \geq v^2(2) > 2p$. Again, the market cannot clear.

Therefore, the necessary condition for a Walrasian equilibrium is

$$v^2(2) \leq 2p \leq v^1(2)$$

- If $v^1(2) = v^2(2)$, either agent gets both units of y and the other 0.
- If $v^1(2) < v^2(2)$, then it must be that agent 1 gets both units of y. Otherwise (i.e., if agent 2 gets them in equilibrium), then it must be $v^2(2) = 2p < v^1(2)$ and hence agent 1 will demands 2 units of y as well, so demand would exceed supply.

(c) Does a Walrasian equilibrium always exist for such an economy? Either prove that it does or give a counterexample.

Solution. An equilibrium always exists. Any p satisfying $v^2(2) \leq 2p \leq v^1(2)$ is an equilibrium price. For any such p, a corresponding equilibrium allocation is

$$y^1 = 2, \quad x^1 = 20 + p\left(e_y^1 - 2\right), \quad y^2 = 0, \quad x^2 = 20 + pe_y^2$$

(d) If a Walrasian equilibrium exists for such an economy, is it Pareto efficient? Either explain why it is or provide a counterexample.

Solution. Yes. In part (b) we derive the necessary conditions for a Walrasian equilibrium. Compared to the results in part (a), we see that any equilibrium allocation must satisfy the conditions in part (a) and hence is efficient.

(e) Suppose we replace the assumption $v^i(1) = 0$ with $v^i(1) > 0$, keeping all the other assumptions. Will a Walrasian equilibrium now always exist? Either explain why or give a counterexample.

Solution. From (b) we know that if an equilibrium exists then a necessary condition is $v^2(2) \leq 2p \leq v^1(2)$. But if for any p satisfying this condition, we also have $p < v^2(1)$, then agent 2 will definitely demand only 1 unit of y. In addition, if $p > v^1(1)$, then it is impossible for agent 1 to demand only 1 unit of y. If we can find such p, then the market for y can never clear and an equilibrium does not exist. Indeed, such case is possible, for example, when

$$\begin{aligned} &v^1(0) = 0, v^1(1) = 1, v^1(2) = 12, \text{ and} \\ &v^2(0) = 0, v^2(1) = 8, v^2(2) = 10 \end{aligned}$$
