Game Theory Recitation 1: Concepts

Xincheng Qiu*
Peking University

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1 Introduction

1.1 History

Game theory as a unique field is often attributed to John von Neumann (1928) and his following book *Theory of Games and Economic Behavior* with Oscar Morgenstern (1944). Their work primarily focused on cooperative game theory. Instead, we are now mainly studying non-cooperative games in this course, which assumes that players cannot sign binding contracts.

John Nash (1950-51) developed an equilibrium concept now known as Nash equilibrium. After that, Reinhard Selten (1965) refined Nash equilibrium with the solution concept of subgame perfect equilibrium, and John Harsanyi (1967) introduced incomplete information games with the solution concept of Bayesian Nash equilibrium. They three shared the 1994 Nobel prize for their fundamental contributions in the pioneering analysis of equilibria in non-cooperative games. The Nobel prize was then awarded to more game theorists: Mirrlees and Vickrey in 1996, Akerlof, Spence and Stiglitz in 2001, Aumann and Schelling in 2005, Myerson, Hurwitz and Maskin in 2007, Shapley and Roth in 2012, Tirole in 2014, Hart and Holmstrom in 2016.

*If you notice any typo, please drop me a line at xincheng.qiu@gmail.com. Comments are also greatly appreciated. The questions are designed for Problem Set 1 by Prof. Xi Weng at Guanghua School of Management, Peking University.

1.2 Why Game Theory

Game theory focuses on strategic interactions among sophisticated individuals.[1] Recall that in price theory, consumers maximize utility subject to budget constraints, and firms maximize profit given technology and factor prices. In many cases, however, one's decision alone is far from sufficient; instead, one's behavior affects and is affected by behaviors of others directly. Game theory is the language to describe and the tool to analyze such interactions.

Game theory states how people behave in a certain reasonable[2] sense. It provides one angle to look at complex real-world problems. It is an art to apply game theory to real-world problems: knowing game theory does not guarantee winning, but it can help understand strategic interactions.[3] Game theory uses some mathematics, but only calculus, probability, and logic (in this course); Strategic thinking in interactions is much more important.

2 Basic Concepts

Definition 1. Normal Form Games.

A normal form game is the collection of strategy sets for each player and their payoff functions $G = \{S_1, \ldots, S_n, u_1, \ldots, u_n\}$.

Note 1. Three key elements of the normal form of a given game:

1. A set of players: $i = 1, 2, ..., n$. Here $n \geq 2$.

2. Strategy space: S_i is a nonempty set, $i = 1, 2, ..., n$.

3. Payoff functions: $u_i : \prod_{k=1}^{n} S_k \to \mathbb{R}$, $i = 1, 2, ..., n$.

Equivalently, a normal form game is a vector-valued function $u : \prod_{k=1}^{n} S_k \to \mathbb{R}^n$.

Definition 2. Extensive Form Games.

We can use a game tree to describe the extensive form of a given game. [4]

Note 2. An *information set* for a player is a collection of decision nodes, where he could not tell which one he is standing on when making decisions. A game has *perfect information* if all information sets are singletons. A *strategy* for a player in an extensive form game is a list of choices for every information set of that player.

[1] You may ask, what does the word "sophisticated" precisely mean here? We will provide several different meanings of being "sophisticated" in several solution concepts.

[2] Again, what does the word "reasonable" exactly mean? It will have different interpretations under various solution concepts.

[3] Recall the game we played in class to guess the two thirds of the average.

[4] We focus on games with *perfect recall*, i.e, every player always remembers what he knew and what he did previously.

Remark 1. A strategy for a player is a mapping from all his information sets. It can be interpreted as a complete and contingent plan covering every circumstance. Why should a player consider the circumstance that will not arise in a certain strategy profile? Imagine that the players are worried about the possibility of mistakes in implementation and thus want to capture all cases.

Definition 3. Mixed Strategies.

We can extend the notion of strategy by allowing players to choose randomly. For player i, a mixed strategy of σ_i is a probability distribution over the set of pure strategies S_i.[5]

Remark 2. It seems natural to use normal forms to describe static games and extensive forms to describe dynamic games. (Here, "dynamic" or "static" is not really in the sense of time, but of information.) However, it is not a must to do so. In fact, we can do it in an opposite way.

How can we introduce mixed strategies in extensive form games? One natural way is to define mixed strategies on the normal form representation of the extensive form game. It turns out that, however, there is a simpler way to define mixed strategies directly on each information set of the extensive form game.

Definition 4. Behavior Strategies.

A behavioral strategy for a player in an extensive form is a list of probability distributions, one for every information set of that player; each probability distribution is over the set of choices at the corresponding information set.

Theorem 1. *Kuhn's Theorem (1953).*

In extensive forms with perfect recall, every mixed strategy has a realization equivalent behavior strategy.

Two strategies for player i are realization equivalent if, fixing the strategies of the other players, the two strategies induce the same distribution over outcomes (terminal nodes). Note that the theorem might fail to hold without perfect recall. Since this course focuses on extensive form games with perfect recall, Kuhn's theorem allows us to restrict attention to behavioral strategies.

Question 1. *Consider the extensive form game showed in Figure 1. What is the normal form of this game?*

[5] We focus on uncorrelated mixed strategies.

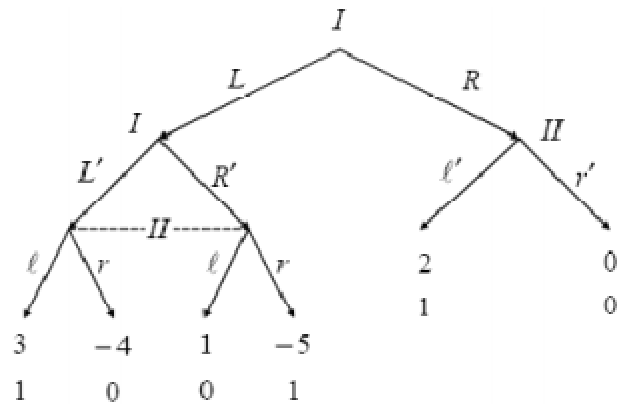

Figure 1: An Extensive Form Game

Solution. The strategy set for I is $\{LL', LR', RL', RR'\}$, and the strategy set for II is $\{ll', lr', rl', rr'\}$. (Remember that a strategy must specify actions in every information set.) The normal form representation of this game is

		player II			
		ll'	*lr'*	*rl'*	*rr'*
player I	*LL'*	3, 1	3, 1	-4, 0	-4, 0
	LR'	1, 0	1, 0	-5, 1	-5, 1
	RL'	2, 1	0, 0	2, 1	0, 0
	RR'	2, 1	0, 0	2, 1	0, 0

RL' and RR' are strategically equivalent. So the *reduced normal form* of this game is

		player II			
		ll'	*lr'*	*rl'*	*rr'*
player I	*LL'*	3, 1	3, 1	-4, 0	-4, 0
	LR'	1, 0	1, 0	-5, 1	-5, 1
	R	2, 1	0, 0	2, 1	0, 0

3 Solution Concepts

Note 3. The equilibrium solution is a strategy profile. It is different from the equilibrium outcome or payoff vector which is the sequence of actual actions.

3.1 Dominant Strategy Equilibrium

Definition 5. Strict (Weak) Domination and Dominance.

1. s_i *strictly dominates* $\hat{s}_i$ for player i if for *every* strategy s_{-i} of the other players,

$$u_i(s_i, s_{-i}) > u_i(\hat{s}_i, s_{-i})$$

   If s_i strictly dominates *all* other strategies of player i, then we say s_i is a *strictly dominant* strategy for player i.

2. s_i *weakly dominates* $\hat{s}_i$ for player i if for *every* strategy s_{-i} of the other players,

$$u_i(s_i, s_{-i}) \geq u_i(\hat{s}_i, s_{-i})$$

   and there exists $s'_{-i} \in S_{-i}$ such that

$$u_i\left(s_i, s'_{-i}\right) > u_i\left(\hat{s}_i, s'_{-i}\right)$$

   If s_i weakly dominates *all* other strategies of player i, then we says_i is a *weakly dominant* strategy for player i.

Criterion 1. *Strictly Dominant Strategy.*

In a strictly dominant strategy equilibrium, every player plays a strictly dominant strategy. We can define a weakly dominant strategy equilibrium similarly.

Remark 3. A strictly dominant strategy is the best thing to do. When a player has a strictly dominant strategy, it would be irrational for him to choose any other strategy, since he would be worse no matter what the other players do. When there is a dominant strategy for every player, the game is dominance solvable under the "individual rationality" assumption. This solution concept only requires each player to be individually rational.

Example 1. Prisoner's Dilemma.

The Prisoner's Dilemma game is often used to illustrate a conflict between individual rationality and collective efficiency.

Example 2. Second-Price Auction.

In a second-price auction, it is a weakly dominant strategy for every player to bid his true valuation (Vickrey, 1961). Truthful revelation is a weakly dominant strategy for every player in the pivotal mechanism (Clarke, 1971).[6]

[6] It is generalized as the Vickrey-Clarke-Groves Mechanism, and second-price auction is a special case of it.

Example 3. Median Voter Theorem.

Consider an election with politicians A and B. Voters' preferences are distributed on $[0,1]$ with a CDF F.[7] A and B simultaneously choose their stances from $[0,1]$ and each voter then votes for the politician whose stance is closer to his preference. The candidate with more votes wins. Assume that each candidate wins with the probability of 0.5 if they choose the same stance. Define x^* as the median voter such that $F(x^*) = 1 - F(x^*) = 0.5$. Median Voter Theorem says that it is weakly dominant for each candidate to choose x^*.

Question 2. *Consider the following modification of a two-bidder second-price sealed-bid auction. Bidder 2 receives an advantage as follows: If bidder 2's bid is at least 80% of bidder 1's bid, then bidder 2 wins and pays 80% of bidder 1's bid. If bidder 2's bid is less than 80% of bidder 1's bid, then bidder 1 wins and pays 1.25 times bidder 2's bid. Suppose bidder i values the object being sold at $v_i, i = 1, 2$. Prove that it is a dominant strategy for each bidder to bid his or her valuation. How would your answer change if bidder 1 paid 1.3 times bidder 2's bid, when bidder 1 wins (but the other rules of the auction are unchanged)?*

Solution. To make it clear, let's first write explicitly the three key elements of the game.

1. Players: bidder 1 and bidder 2.
2. Strategy sets: $S_1 = S_2 = \mathbb{R}_+$.
3. Payoff Functions:

$$U_1(b_1, b_2) = \begin{cases} 0 & b_2 \geq 0.8b_1 \\ v_1 - 1.25b_2 & b_2 < 0.8b_1 \end{cases}$$

$$U_2(b_1, b_2) = \begin{cases} v_2 - 0.8b_1 & b_2 \geq 0.8b_1 \\ 0 & b_2 < 0.8b_1 \end{cases}$$

We can prove that bidding v_1 is a weakly dominant strategy for bidder 1 (and the proof for bidder 2 is similar). There are two cases.

Case 1: $b_2 < 0.8v_1$. Then $U_1(v_1, b_2) = v_1 - 1.25b_2 \geq U_1(b_1, b_2)$.

Case 2: $b_2 \geq 0.8v_1$. Then $U_1(v_1, b_2) = 0 \geq U_1(b_1, b_2)$.

Thus bidding v_1 is optimal. Bidding v_1 also weakly dominates every other bid and hence v_1 is weakly dominant. Suppose $b_1 < v_1$, there exists $b_2 \in (0.8b_1, 0.8v_1)$, with $U_1(b_1, b_2) = 0 < v_1 - 1.25b_2 = U_1(v_1, b_2)$. Suppose $b_1 > v_1$, there exists $b_2 \in (0.8v_1, 0.8b_1)$, with $U_1(b_1, b_2) = v_1 - 1.25b_2 < 0 = U_1(v_1, b_2)$.

[7] For example, we can interpret 1 as the most "liberal" and 0 as the most "conservative."

But when the setup changes, the payoff function for bidder 1 changes:

$$U_1(b_1, b_2) = \begin{cases} 0 & b_2 \geq 0.8b_1 \\ v_1 - 1.3b_2 & b_2 < 0.8b_1 \end{cases}$$

It is still a weakly dominant strategy for bidder 2 to bid his valuation, since his payoff function is unchanged and dominance is irrelevant of others' payoff functions. It is *not* a dominant strategy for bidder 1 to bid his valuation now. To illustrate, consider when b_2 satisfies that $1.25b_2 < v_1 < 1.3b_2$. Now bidding v_1 makes bidder 1 the winner but yields a payoff of $v_1 - 1.3s_2 < 0$. Bidder 1 prefers to lose, which yields a payoff of 0. To generalize this, if the payoff function is

$$U_1(b_1, b_2) = \begin{cases} 0 & b_2 \geq \alpha b_1 \\ v_1 - \beta b_2 & b_2 < \alpha b_1 \end{cases}$$

We can prove that the weakly dominant strategy is $b_1^* = \frac{1}{\alpha\beta} v_1$.

Case 1: $b_2 < \alpha b_1^*$. Then $U_1(b_1^*, b_2) = v_1 - \beta b_2 \geq U_1(b_1, b_2)$.

Case 2: $b_2 \geq \alpha b_1^*$. Then $U_1(b_1^*, b_2) = 0 \geq U_1(b_1, b_2)$.

Thus bidding $b_1^* = \frac{1}{\alpha\beta} v_1$ is optimal. Bidding $b_1^* = \frac{1}{\alpha\beta} v_1$ also weakly dominates every other bid and hence $b_1^* = \frac{1}{\alpha\beta} v_1$ is weakly dominant. Suppose $b_1 < b_1^*$, there exists $b_2 \in (\alpha b_1, \alpha b_1^*)$, with $U_1(b_1, b_2) = 0 < v_1 - \beta b_2 = U_1(b_1^*, b_2)$. Suppose $b_1 > b_1^*$, there exists $b_2 \in (\alpha b_1^*, \alpha b_1)$, with $U_1(b_1, b_2) = v_1 - \beta b_2 < 0 = U_1(b_1^*, b_2)$.

Therefore, for $b_1^*$to be a weakly dominant strategy, $b_1^* = \frac{1}{\alpha\beta} v_1$. When $\alpha = 0.8$ and $\beta = 1.3$, $b_1^* = \frac{1}{0.8 \times 1.3} v_1 = \frac{25}{26} v_1$ is weakly dominant strategy.

3.2 Iterated Elimination of Dominated Strategies (IEDS)

Unfortunately, strictly (weakly) dominant strategy does not always exist. We can go further if we assume more: players are rational, they know that the other players are also rational, they know that other players also know that he or she is rational, and so forth. This is called the *common knowledge of rationality.*

Definition 6. Dominated Strategy.

If there exists some strategy (including mixed strategies) $\hat{\sigma}_i$ that strictly dominates s_i, we say that s_i is a strictly dominated strategy (and dominated by $\hat{\sigma}_i$). We can define weakly dominated strategy similarly.

Note 4. If a strictly (weakly) dominant strategy exists, it is unique. But dominated strategies are not necessarily unique. The claim that "for some player, strategy x is strictly (weakly) dominated" does not mean every other strategy dominates x. It only means there exists some strategy that dominates x, so you should name it. An analogy: "x is dominated" is analogous to "x is worse," so you should keep in mind x is worse than what; but "strategy y is dominant" is similar to "y is best", that is, y is better than (or at least as good as) all the other strategy.

Criterion 2. *Iterated Elimination of strictly Dominated Strategies (or Rationalizable Actions).*

Remark 4. Intuitively, rational players do not play strictly dominated strategies, for there exists another strategy that is always better. To perform *strict IEDS*, we require that players are rational and there is common knowledge of rationality. The solution concept that survives iterated elimination of strictly dominated strategies is also called *rationalizability* (i.e., repeatedly remove all actions which are never a best reply). For each player i, a rationalizable action a_i is a best response to certain strategies of other players.

Remark 5. In finite games, the order in which strictly dominated strategies are eliminated is irrelevant.We can similarly define *Iterated Elimination of weakly Dominated Strategies.* However, this procedure has to be dealt with very carefully, since the order of elimination matters. In order to avoid this problem, it is required to identify all the strategies that are weakly dominated for every player at every step, and then eliminate all such strategies in this step.

Example 4. Consider the game shown in the table.

		Player 2	
		L	R
Player 1	A	4, 0	0, 0
	T	3, 2	2, 2
	M	1, 1	0, 0
	B	0, 0	1, 1

Order 1: $M \to L \to A \to B$. Then we are left with (T, R);

Order 2: $B \to R \to T \to M$. Then we are left with (A, L);

The correct order: eliminate M, B at the same time and we are left with (AT, LR). So the game does not have an iterated weak dominant-strategy equilibrium.

Remark 6. A rational player may not exclude playing a weakly dominated strategy. It is possible that a weakly dominated strategy is played in a Nash equilibrium.

Question 3. *An election has three candidates, A, B, C, and three voters,* $i = 1, 2, 3$. *The voting rule is such that: The elected candidate is the one chosen by voter 2 and 3 if they vote for the same candidate, and the one chosen by voter 1 otherwise. Suppose that* $u_1(A) > u_1(B) > u_1(C)$, $u_2(C) > u_2(A) > u_2(B)$ *and* $u_3(B) > u_3(C) > u_3(A)$. *Find the unique outcome implied by iterated elimination of dominated strategies.*

Solution. In this setting, we only care about the ranking and the number of the utility itself is meaningless. For simplicity, we can assign numbers for the utility functions. Let $u_1(A) = 3, u_1(B) = 2, u_1(C) = 1$; $u_2(A) = 2, u_2(B) = 1, u_2(C) = 3$; $u_3(A) = 1, u_3(B) = 3, u_3(C) = 2$.

Note that for voter 1, voting for A is a weakly dominant strategy. To see this, consider first when voter 2 and voter 3 vote for different candidates, now the one chosen by voter 1 will be elected so voter 1 prefers voting for A. Consider then when voter 2 and voter 3 vote for the same candidate, now voter 1 will not affect the outcome. Therefore, voting for A is a weakly dominant strategy for voter 1. In other words, for voter 1, voting for B and voting for C are weakly dominated by voting for A and thus should be eliminated. Besides, B is a weakly dominated strategy for voter 2, and A is a weakly dominated strategy for voter 3.

First, for voter 1, eliminate the strategies of voting for B and voting for C; for voter 2, eliminate B; for voter 3, eliminate A.

Voter 1 chooses A

Voter 2 \ Voter 3	B	C
A	A (3, 2, 1)	A (3, 2, 1)
C	A (3, 2, 1)	C (1, 3, 2)

Second, for voter 2, eliminate A; for voter 3, eliminate B.

Voter 1 chooses A

Voter 2 \ Voter 3	C
C	C (1, 3, 2)

The unique strategy profile that survives weak IEDS is (A, C, C), and the corresponding outcome is candidate C being chosen.

3.3 Nash Equilibrium

3.3.1 Definition

Unfortunately, the set of rationalizable actions may still be too large.

Criterion 3. *Nash Equilibrium. The strategy profile σ^* is a Nash Equilibrium if for all i and all $s_i \in S_i$,*

$$u_i\left(\sigma_i^*, \sigma_{-i}^*\right) \geq u_i\left(s_i, \sigma_{-i}^*\right)$$

Remark 7. Nash equilibrium is basically based on two assumptions:

1. each player is playing his best response given his belief about what the other players will play;

2. these beliefs are correct in the sense that the beliefs are consistent with actual actions.

"*Mutual best responses*" and "*no deviation*" are the usual interpretations, which may help a lot in solving problems and finding Nash equilibria, but it is far from enough for you to understand the great concept of Nash equilibrium without catching the importance of belief.

The first assumption is rather acceptable, but the consistency of beliefs with actual behaviors is a bit strong. There are several justifications for Nash equilibrium, for example, a self-enforcing agreement, a viable recommendation, or a non-deviation state that players can converge to by learning or evolution, etc.

Theorem 2. *Existence of Nash Equilibrium. (Nash, 1951)*

Every finite normal form game has at least one Nash equilibrium in mixed strategies.

We will not prove it here. The main idea is that a Nash equilibrium is mathematically a fixed point of the best reply correspondence.The existence of Nash equilibria thus boils down to the existence of fixed points. Then apply Kakutani's fixed point theorem.

Note 5. How to find (mixed strategy) Nash equilibria?

1. In a Nash equilibrium, a pure strategy that is strictly dominated (by a pure or mixed strategy) can never be played with positive probability. Therefore, we can first apply strict IEDS and focus on the reduced game.

2. Indifference Principle. The only reason for the player to randomize these strategies is that they generate same payoff given the others' strategy profile. However, it is a necessary but not sufficient condition for a Nash equilibrium. See the game below. The mixed strategy profile $\left(\frac{1}{2}, \frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}\right)$satisfies the indifference principle but is not a Nash equilibrium. (Player 1 can get better off by choosing D with certainty.)

		Player 2	
		L	R
Player 1	U	3, 0	0, 2
	M	0, 2	3, 0
	D	2, 0	2, 1

3. In this case, go back to the definition of Nash equilibrium. Calculate the best reply correspondence and look for mutual best responses.

3.3.2 Relationship between IEDS and NE

Proposition 1. *A Nash equilibrium survives strict IEDS. If strict IEDS leads to a unique outcome, then it is a Nash equilibrium.*

Remark 8. If we can perform strict IEDS, all Nash equilibria will survive in the reduced game. So please always use strict IEDS to simplify the game at first whenever possible. This is useful in finding Nash equilibria in "large" games.

Proposition 2. *If weak IEDS leads to a unique outcome, then it is a Nash equilibrium; but it may not be the unique NE. Weak IEDS may eliminate some Nash equilibria.*

Example 5. Weak IEDS and Nash Equilibrium.

		Player 2	
		L	R
Player 1	U	1, 1	0, 0
	D	0, 0	0, 0

By performing weak IEDS, the outcome is (U, L). It is a Nash equilibrium but not the unique one. In fact, (D, R) is also Nash equilibrium. The question is, why would people ever play(D, R) instead of (U, L)?

Criterion 4. *Perfect Equilibrium.*

An equilibrium σ of a finite normal from game G is (trembling hand) perfect if there exists a sequence $\left\{\sigma^k\right\}_k$ of completely mixed strategy profiles converging to σ such that σ_i is a best reply to every σ_{-i}^k in the sequence.

Proposition 3. *Every perfect equilibrium is a Nash equilibrium in weakly undominated strategies.*

Question 4. *Can you find other Nash equilibrium in Question 3?*

Solution. The result that survives the weak IEDS (A, C, C) is one Nash equilibrium. But weak IEDS cannot guarantee that it is the unique Nash equilibrium. To find all Nash equilibrium, we need to reconsider this game in three 3×3 tables.

Voter 1 chooses A

Voter 2 \ Voter 3	A	B	C
A	**A (3, 2, 1)**	**A (3, 2, 1)**	A (3, 2, 1)
B	A (3, 2, 1)	B (2, 1, 3)	A (3, 2, 1)
C	A (3, 2, 1)	A (3, 2, 1)	**C (1, 3, 2)**

Voter 1 chooses B

Voter 2 \ Voter 3	A	B	C
A	A (3, 2, 1)	B (2, 1, 3)	B (2, 1, 3)
B	B (2, 1, 3)	**B (2, 1, 3)**	B (2, 1, 3)
C	B (2, 1, 3)	B (2, 1, 3)	C (1, 3, 2)

Voter 1 chooses C

Voter 2 \ Voter 3	A	B	C
A	A (3, 2, 1)	C (1, 3, 2)	C (1, 3, 2)
B	C (1, 3, 2)	B (2, 1, 3)	C (1, 3, 2)
C	C (1, 3, 2)	C (1, 3, 2)	**C (1, 3, 2)**

There are 5 Nash equilibria in this game: (A, A, A), (A, A, B), (A, C, C), (B, B, B), and (C, C, C).

Question 5. *Describe the pure strategy Nash equilibrium strategies and outcomes of the game in Question 1.*

Solution. We can see from the answer in Question 1:

player I \ player II	*ll'*	*lr'*	*rl'*	*rr'*
LL'	**3, 1**	**3, 1**	-4, 0	-4, 0
LR'	1, 0	1, 0	-5, 1	-5, 1
R	2, 1	0, 0	**2, 1**	0, 0

The pure strategy Nash equilibria are (LL', ll'), (LL', lr') and (R, rl').

3.4 Refinements

As we already see, many games have multiple equilibria and they are not equivalent. Which one is more reasonable? Game theorists develop various equilibrium refinements.

Criterion 5. *Subgame Perfection.*

The strategy profile s is a subgame perfect equilibrium if s prescribes a Nash equilibrium in every subgame.

Note 6. How can we find subgame perfect equilibria?

1. One natural way to find subgame perfect equilibria is to follow the definition. First, find all Nash equilibria. Second, check for each of them if it is still a Nash equilibrium in every subgame. You can imagine how tedious it is to do so, especially when the game has many subgames and many Nash equilibria.

2. A quicker way is to apply the *backward induction.* First, start with a minimal subgame and select a Nash equilibrium. Second, replace the selected subgame with the payoff vector associated with the selected Nash equilibrium, and take notes of the strategies. Now we have a smaller extensive form game. Third, repeat the above two steps, and in the end obtain one subgame perfect equilibrium. Repeat the procedure by choosing a different Nash equilibrium in some step and thus obtain a different subgame perfect equilibrium, and so on.

Question 6. *Describe the pure strategy subgame perfect equilibria (there may only be one) in Question 1.*

Solution. By backward induction (Figure 2),

the only subgame perfect equilibrium is (LL', ll').

Remark 9. Subgame perfect equilibrium is a refinement of Nash equilibrium. It eliminates some unreasonable Nash equilibria that involve "incredible threats." However, some subgame perfect equilibrium will still be not so reasonable in some sense. Selten's Horse (Figure 3) provides an example.

Example 6. Selten's Horse.

Selten's Horse game has only one subgame – itself. So the set of subgame perfect equilibria in this game is exactly the same as that of Nash equilibria.

Player *I* chooses *A*

		Player *III*	
		L	*R*
Player *II*	*a*	$\underline{\mathbf{1}}, \underline{\mathbf{1}}, \underline{\mathbf{1}}$	$1, 1, \underline{1}$
	d	$\underline{0}, 0, \underline{1}$	$\underline{4}, \underline{4}, 0$

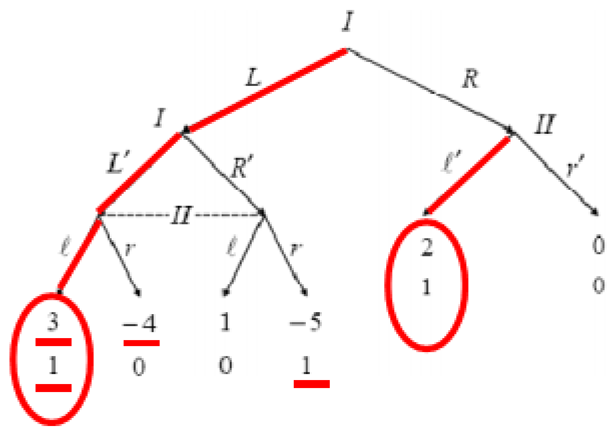

Figure 2: Backward Induction

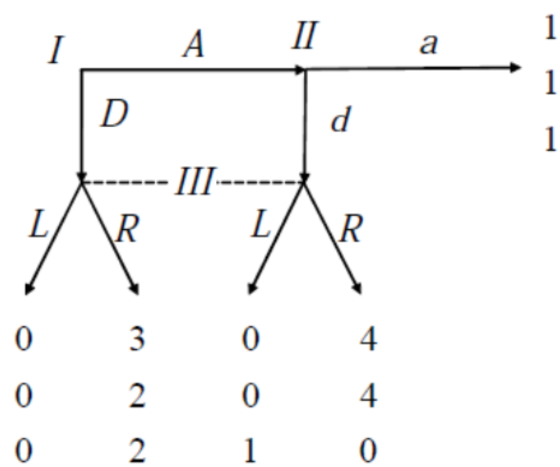

Figure 3: Selten's Horse

Player I chooses D — Player III

		L	R
Player II	a	0, $\underline{0}$, 0	**$\underline{3}$, $\underline{2}$, $\underline{2}$**
	d	$\underline{0}$, $\underline{0}$, 0	3, $\underline{2}$, $\underline{2}$

The pure strategy Nash equilibria and thus subgame perfect equilibria are $(A, a, L), (D, a, R)$. Consider the second Nash equilibrium (D, a, R). Player II's plan to play a is rational only in the very limited sense that, given that Player I plays D, what Player II plays does not affect the outcome. However, if Player II actually finds himself having to make a decision, d seems to be a better strategy than a, since given that Player III plays R, d would give Player II a payoff of 4, while a would only give a payoff of 1. This example addresses the problem that even subgame perfect equilibria could be "unreasonable" in some sense. We will still need some other refinements.

4 Summary

Belief is an important concept in game theory, which will be discussed further in this course. An optimal response is based on the player's belief. We can define a player to be rational if his chosen strategy is a best response to his belief about what the opponent will do. Try to understand the reasonableness and the structure of beliefs behind each solution concept. Let's see some examples we have discussed above.

- Strictly/weakly dominant strategy equilibrium: Beliefs are not needed here; individual rationality alone is enough.

- Strict IEDS: Rationality alone is insufficient here; common knowledge of rationality is needed. That is, I will not play strictly dominated strategies, and I believe you will not play strictly dominated strategies, and I believe that you believe that I will not play strictly dominated strategies, and so forth.

- Nash equilibrium: The beliefs are correct in the sense that one's belief is consistent with the others' actual behavior. Do not simply interpret a Nash equilibrium as a non-deviation state; the reason why there is no deviation is essentially because the beliefs are correct and realized (and they are optimizing according to such beliefs).

5 Exercises

Question 7. *Two firms are competing in the market for widgets. The output of the two firms are perfect substitutes, so that the demand curve is given by* $Q = \max\{\alpha - p, 0\}$*, where* p *is low price. Firm* i *has constant marginal cost of production* $0 < c_i < \alpha$*, and no capacity constraints. Firms simultaneously announce prices, and the lowest pricing firm has sales equal to total market demand. The division of the market in the event of a tie (i.e., both firms announcing the same price) depends upon their costs: if firms have equal costs, then the market demand is evenly split between the two firms; if firms have different costs, the lowest cost firm has sales equal to total market demand, and the high cost firm has no sales.*

1. *Suppose* $c_1 = c_2 = c$ *(i.e., the two firms have identical costs). Restricting attention to pure strategies, prove that there is a unique Nash equilibrium. What is it? What are firm profits in this equilibrium?*

2. *Suppose* $c_1 < c_2 < \frac{\alpha + c_1}{2}$*. Still restricting attention to pure strategies, describe the set of Nash equilibria. Are there any in weakly undominated strategies?*

3. *We now add an investment stage before the pricing game. At the start of the game, both firms have identical costs of* c_H*, but before the firms announce prices, firm 1 has the opportunity to invest in a technology that gives a lower unit cost* c_L *of production (where* $c_L < c_H < \frac{\alpha + c_L}{2}$*). This technology requires an investment of* $k > 0$*. The acquisition of the technology is public before the pricing game subgame is played. Describe the extensive form of the game. Describe a subgame perfect equilibrium in which firm 1 acquires the technology (as usual, make clear any assumptions you need to make on the parameters). Is there a subgame perfect equilibrium in which firm 1 does not acquire the technology? If not, why not? If there is, compare to the equilibrium in which firm 1 acquires the technology.*

Solution. 1. When $c_1 = c_2 = c$, the unique pure strategy Nash equilibrium is $p_1 = p_2 = c$ (Bertrand Competition model). To prove this, we start with the best response functions (derived from the utility functions).

$$\pi_1(p_1, p_2) = \begin{cases} \max\{\alpha - p_1, 0\} \cdot (p_1 - c_1) & p_1 < p_2 \\ \frac{1}{2}\max\{\alpha - p_1, 0\} \cdot (p_1 - c_1) & p_1 = p_2 \\ 0 & p_1 > p_2 \end{cases}$$

$$p_1^*(p_2) = \begin{cases} \frac{\alpha+c}{2}, & p_2 > \frac{\alpha+c}{2} \\ p_2 - \varepsilon & c < p_2 \leq \frac{\alpha+c}{2} \\ [p_2, +\infty] & p_2 = c \\ (p_2, +\infty] & p_2 < c \end{cases}$$

Here ε is a positive number that is very close to 0. Similarly, we can write the best response for firm 2 as a function of firm 1's price $p_2^*(p_1)$. The best responses intersect at (c, c) on the $p_1 - p_2$ plain.

2. Again, first write the best responses (derived from the utility functions).

$$\pi_1(p_1, p_2) = \begin{cases} \max\{\alpha - p_1, 0\} \cdot (p_1 - c_1) & p_1 \leq p_2 \\ 0 & p_1 > p_2 \end{cases}$$

$$p_1^*(p_2) = \begin{cases} \frac{\alpha+c_1}{2}, & p_2 > \frac{\alpha+c_1}{2} \\ p_2 & c_1 < p_2 \leq \frac{\alpha+c_1}{2} \\ [p_2, +\infty] & p_2 = c_1 \\ (p_2, +\infty] & p_2 < c_1 \end{cases}$$

Similarly,

$$p_2^*(p_1) = \begin{cases} \frac{\alpha+c_2}{2}, & p_1 > \frac{\alpha+c_2}{2} \\ p_1 - \varepsilon & c_2 < p_1 \leq \frac{\alpha+c_2}{2} \\ [p_1, +\infty] & p_1 \leq c_2 \end{cases}$$

The best responses intersect at $\{(p_1, p_2) : c_1 \leq p_1 = p_2 \leq c_2\}$, which is the set of Nash equilibria.

None of the equilibria are in weakly undominated strategies. First, $p_2 < c_2$ are weakly dominated strategies by $p_2 = c_2$ for firm 2. Then, $p_2 = c_2$ is weakly dominated by any $p_2 > c_2$.

3. If firm 1 does not invest, the subsequent subgame is the same as in (1). The Nash equilibrium in subgame (1) yields a payoff vector $(0, 0)$. If firm 1 invests, the subsequent subgame becomes (2), with the Nash equilibrium yielding a payoff vector $((\alpha - p)(p - c_L) - k, 0)$, where $c_L \leq p \leq c_H < \frac{\alpha+c_L}{2}$. Therefore $(\alpha - p)(p - c_L)_{\max} = (\alpha - c_H)(c_H - c_L)$.

4. if $k < (\alpha - c_H)(c_H - c_L)$, there exists subgame perfect equilibria, in which firm 1 chooses to invest, and in the second stage $\{(p_1, p_2) : p_0 \leq p_1 = p_2 \leq c_H\}$ where $(\alpha - p_0)(p_0 - c_L) = k$.

   (a) if $k = (\alpha - c_H)(c_H - c_L)$, there two subgame perfect equilibria: firm 1 invests, $p_1 = p_2 = c_H$; firm 2 does not invest, $p_1 = p_2 = c_H$;

   (b) if $k > (\alpha - c_H)(c_H - c_L)$, in the subgame perfect equilibria, firm 1 does not invest and $p_1 = p_2 = c_H$.

Question 8. *Consider the following game G between two players. Player 1 first chooses between A or B, with A giving payoff of 1 to each player, and B giving a payoff of 0 to player 1 and 3 to player 2. After player 1 has publicly chosen between A and B, the two players play the following simultaneous move game (with 1 being the row player)*

		Player 2	
		L	R
Player 1	U	*1, 1*	*0, 0*
	D	*0, 0*	*3, 3*

1. *Write down the extensive form of G, and find all pure strategy subgame perfect equilibria.*

2. *Write down the normal form of G.*

3. *In the above normal form game, what is the result of the iterated deletion of weakly dominated strategies? Discuss your finding using the concept of forward induction.*

Solution. 1. Four pure strategy subgame perfect equilibria:

$$\{AUU, LL\}, \{BUD, LR\}, \{ADU, RL\}, \{ADD, RR\}$$

2. Reduced normal form:

		Player 2			
		LL	LR	RL	RR
Player 1	AU	2, 2	2, 2	1, 1	1, 1
	AD	1, 1	1, 1	4, 4	4, 4
	BU	1, 4	0, 3	1, 4	0, 3
	BD	0, 3	3, 6	0, 3	3, 6

3. Perform weak IEDS: $BU \to LL, RL \to AU \to LR \to BD \to (AD, RR)$. Forward induction: player 2 infers that the only reason player 1 did not play B is that he intends to play D in the coordination game.

Game Theory Recitation 2: Repeated Games

Xincheng Qiu[*]

Peking University

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Abstract

In many cases, players interact not just once but repeatedly. Repeated games model is designed to capture the idea of long-term interaction, where a player takes into consideration the effect of his current behavior on the other players' future behavior. Repeated games can help explain cooperation while rationality predicts defection in the prisoner's dilemma.

1 A Motivating Example

In the following Prisoners' Dilemma game, the unique Nash equilibrium is (D, D). However, the outcome (C, C) is Pareto dominant, which illustrates tension between mutual cooperation and self-interests.

	C	D
C	4, 4	0, $\underline{5}$
D	$\underline{5}$, 0	$\underline{1}$, $\underline{1}$

An important observation is that people interact more than once under many real-world circumstances. If the game is repeated for finite times, however, we can argue by backward induction that the equilibrium outcome is still defection (D, D). How can we achieve cooperation?

[*] If you notice any typo, please drop me a line at xincheng.qiu@gmail.com. Comments are also greatly appreciated.

1.1 Augmented Prisoners' Dilemma

Consider an augmented Prisoners' Dilemma with an additional strategy for each player. Play this new stage game is twice and add payoffs.

	C	D	A
C	4, 4	0, $\underline{5}$	0, 0
D	$\underline{5}$, 0	$\underline{1}$, $\underline{1}$	0, 0
A	0, 0	0, 0	$\underline{3}$, $\underline{3}$

There exists a subgame perfect equilibrium where (C, C) can be realized on the equilibrium path: in the first stage, play (C, C); in the second stage, play (A, A) if (C, C) is realized in the first stage, otherwise play (D, D).

1.2 Infinite Repeated Prisoners' Dilemma

Assume that the Prisoners' Dilemma is repeated infinitely. Introduce a discount factor $\delta \in [0,1]$. Consider the trigger strategy: cooperate in the first stage; cooperate if the previous stage outcome is (C, C), otherwise defect forever. For a strategy profile to be subgame perfect, we only need to check subgame after *one deviation*. We can show that trigger strategy is a subgame perfect equilibrium when

$$4 \geq 5(1-\delta) + 1 \cdot \delta \Leftrightarrow \delta \geq \frac{1}{4}$$

2 Repeated Games

Definition 1. A *repeated game* is a dynamic game in which the same static game (*stage game*) is played at every stage.

In a finite repeated game, it is easy to find subgame perfect equilibria by backward induction. However, backward induction cannot be applied to infinite horizon games. So, how do we solve for infinite repeated games?

Theorem 1. *A strategy profile is subgame perfect iff there are no profitable one-shot deviations.*

Payoffs of $G(\infty)$ are

$$U_i(s) = (1-\delta) \sum_{t=0}^{\infty} \delta^t u_i\left(a^t(s)\right)$$

Every strategy profile can be represented by an *automaton* $(\mathcal{W}, w^0, f, \tau)$, where

- $\mathcal{W}$ is set of states,
- w^0 is initial state,
- $f : \mathcal{W} \to A$ is output function (decision rule), and
- $\tau : \mathcal{W} \times A \to \mathcal{W}$ is transition function.

Let $V_i(w)$ be i's value from being in the state $w \in \mathcal{W}$, i.e.,

$$V_i(w) = (1 - \delta) u_i(f(w)) + \delta V_i(\tau(w, f(w)))$$

Definition 2. Player i has a *profitable one-shot deviation* if there is some action a_i such that

$$(1 - \delta) u_i(a_i, f_{-i}(w)) + \delta V_i(\tau(w, (a_i, f_{-i}(w)))) > V_i(w)$$

Corollary 1. *A strategy profile is subgame perfect iff* $\forall w \in \mathcal{W}$, $f(w)$ *is a Nash equilibrium of the normal form game*

$$g_i^w(a) = (1 - \delta) u_i(a) + \delta V_i(\tau(w, a))$$

3 Exercises[1]

Exercise 1. Consider the following stage game.

	L	C	R
U	5, 5	7, 0	3, x
M	0, 0	4, 1	0, 0
D	0, 0	0, 0	0, 0

1. Suppose $x = 6$ and the game is repeated infinitely with perfect monitoring. Both players have a discount factor δ. Describe a strategy profile such that the two players play (U, L) in each period on the equilibrium path. How large δ needs to be so that the strategy profile you have just defined is a subgame perfect equilibrium?

2. Suppose $x = 0$ and consider the following behavior in the infinitely repeated game with perfect monitoring: Play MC in period $t = 0$. Play MC as long as no one has

[1] The questions are designed for Problem Set 2 by Prof. Xi Weng at Guanghua School of Management, Peking University.

deviated in the previous two periods. If any player deviates, play DR for two periods and then return to MC. For what values of the common discount factor δ is this profile a subgame perfect equilibrium of the infinitely repeated game?

3. Suppose $x = 0$ and now the payoff from the action profile DL is (0, 1). How does this change to the stage game affect the range of discount factors for which the profile in (2) is a subgame perfect equilibrium of the infinitely repeated game?

Solution. 1. Player 1 has no incentive to deviate from (U, L) but player 2 has an incentive to deviate to R. To avoid player 2's deviation, player 1 has to punish player 2 by playing another strategy if he deviates. Note that U is the strictly dominant strategy for player 1, so the subgame perfect equilibrium cannot be supported by any trigger strategy. Consider the strategy where defection can be forgiven: At period 0, they cooperate, i.e., play (U, L); If cooperation is not achieved in period $t-1$, the punishment (D, R) is imposed in period t; if the punishment has not been carried out, they continue to play (D, R); they revive cooperation until the punishment has been implemented in the preceding period.

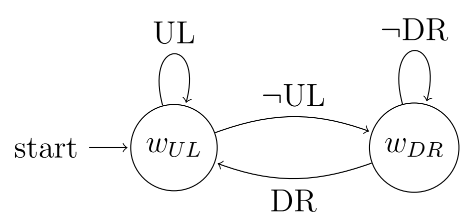
According to one deviation principle,

$$5 \geq 6(1-\delta) + 0 + 5\delta^2 \Leftrightarrow (5\delta - 1)(\delta - 1) \leq 0 \Rightarrow \delta \geq \frac{1}{5}$$

$$0 + 5\delta \geq 3(1-\delta) + 0 + 5\delta^2 \Leftrightarrow (5\delta - 3)(\delta - 1) \leq 0 \Rightarrow \delta \geq \frac{3}{5}$$

For this strategy to be a SPE, we need $\delta \geq \frac{3}{5}$.

An equivalent and illustrating way to derive the conditions is to exploit automata.



$$\begin{cases} V_i(w_{UL}) = 5(1-\delta) + \delta V_i(w_{UL}) \\ V_i(w_{DR}) = 0 + \delta V_i(w_{UL}) \end{cases}$$

So we have $V_i(w_{UL}) = 5$, $V_i(w_{DR}) = 5\delta$. The normal form associated with w_{UL} is

	L	C	R
U	$5,\ 5$	$7(1-\delta)+5\delta^2,\ 5\delta^2$	$3(1-\delta)+5\delta^2,\ 6(1-\delta)+5\delta^2$
M	$5\delta^2,\ 5\delta^2$	$4(1-\delta)+5\delta^2,\ (1-\delta)+5\delta^2$	$5\delta^2,\ 5\delta^2$
D	$5\delta^2,\ 5\delta^2$	$5\delta^2,\ 5\delta^2$	$5\delta^2,\ 5\delta^2$

Note that $0 \leq \delta \leq 1$, so for UL to be a NE, we must have $5 \geq 6(1-\delta)+5\delta^2$, i.e., $\delta \geq \frac{1}{5}$.

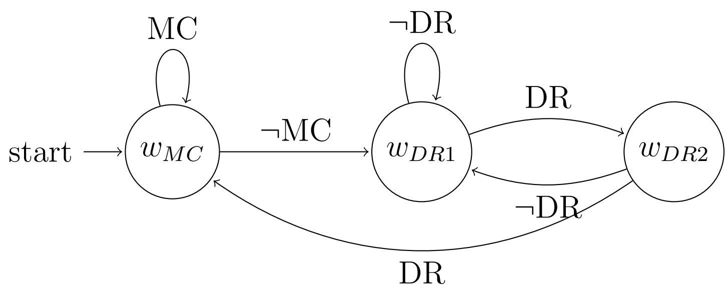
The normal form associated with w_{DR} is

	L	C	R
U	$5(1-\delta)+5\delta^2,\ 5(1-\delta)+5\delta^2$	$7(1-\delta)+5\delta^2,\ 5\delta^2$	$3(1-\delta)+5\delta^2,\ 6(1-\delta)+5\delta^2$
M	$5\delta^2,\ 5\delta^2$	$4(1-\delta)+5\delta^2,\ (1-\delta)+5\delta^2$	$5\delta^2,\ 5\delta^2$
D	$5\delta^2,\ 5\delta^2$	$5\delta^2,\ 5\delta^2$	$5\delta,\ 5\delta$

For DR to be a NE, we must have $5\delta \geq 3(1-\delta)+5\delta^2$, i.e., $\delta \geq \frac{3}{5}$.

Therefore, for such strategy to be a SPE, we need to have $\delta \geq \frac{3}{5}$. Other potential strategies are left for you to have a brain storm.

2. The strategy profile can be represented by the following automaton:



$$\begin{cases} V_i(w_{MC}) = (1-\delta)\,u_i(MC) + \delta V_i(w_{MC}) \\ V_i(w_{DR1}) = 0 + \delta V_i(w_{DR2}) \\ V_i(w_{DR2}) = 0 + \delta V_i(w_{MC}) \end{cases}$$

It solves $V_1(w_{MC}) = 4$, $V_1(w_{DR1}) = 4\delta^2$, $V_1(w_{DR2}) = 4\delta$ and $V_2(w_{MC}) = 1$, $V_2(w_{DR1}) = \delta^2$, $V_2(w_{DR2}) = \delta$. The normal form associated with w_{MC} is[2]

	L	C	R
U	/	$7(1-\delta)+4\delta^3,\ \delta^3$	/
M	$4\delta^3,\ \delta^3$	$4,\ 1$	$4\delta^3,\ \delta^3$
D	/	$4\delta^3,\ \delta^3$	/

[2]It suffices to check unilateral deviation.

Note that $0 \leq \delta \leq 1$, so for MC to be a NE, we should have

$$4 \geq 7(1-\delta) + 4\delta^3 \Leftrightarrow (\delta - 1)(2\delta + 3)(2\delta - 1) \leq 0 \Rightarrow \delta \geq \frac{1}{2}$$

The normal form associated with w_{DR1} is

	L	C	R
U	/	/	$3(1-\delta) + 4\delta^3$, δ^3
M	/	/	$4\delta^3$, δ^3
D	$4\delta^3$, δ^3	$4\delta^3$, δ^3	$4\delta^2$, δ^2

For DR to be a NE, we should have

$$4\delta^2 \geq 3(1-\delta) + 4\delta^3 \Leftrightarrow \left(4\delta^2 - 3\right)(\delta - 1) \leq 0 \Rightarrow \delta \geq \frac{\sqrt{3}}{2}$$

The normal form associated with w_{DR2} is

	L	C	R
U	/	/	$3(1-\delta) + 4\delta^3$, δ^3
M	/	/	$4\delta^3$, δ^3
D	$4\delta^3$, δ^3	$4\delta^3$, δ^3	4δ, δ

For DR to be a NE, we should have

$$4\delta \geq 3(1-\delta) + 4\delta^3 \Leftrightarrow (\delta - 1)(2\delta + 3)(2\delta - 1) \leq 0 \Rightarrow \delta \geq \frac{1}{2}$$

Therefore, this strategy profile is a SPE when $\delta \geq \frac{\sqrt{3}}{2}$.

3. If the payoff from DL changes to (0, 1), the payoff from DL in the above normal forms becomes $(4\delta^3, 1 - \delta + \delta^3)$. Now we need some *additional* non-deviation conditions

$$\begin{cases} \delta^2 \geq 1 - \delta + \delta^3 \\ \delta \geq 1 - \delta + \delta^3 \end{cases}$$

Note that $\delta \geq \delta^2$ for $0 \leq \delta \leq 1$. We only need to solve for $\delta^2 \geq 1 - \delta + \delta^3$. So $(\delta - 1)^2(\delta + 1) \leq 0$, i.e., $\delta = 1$.

Exercise 2. Consider the following simultaneous move game

	L	M	R
U	1, -1	0, 0	10, 4
M	0, 0	2, 1	7, 5
D	8, 9	5, 10	0, 0

1. Find all the Nash Equilibria (in **both** pure strategies and mixed strategies) of this game.

2. Suppose the game is repeated twice without discounting. Construct a subgame perfect equilibrium such that player 1 plays M and player 2 plays R in the first period.

3. Now suppose the game is repeated infinitely. Both players have a discount factor δ. Define a strategy profile such that the two players play (D, L) in each period on the equilibrium path. How large δ needs to be so that the strategy profile you have just defined is a subgame perfect equilibrium?

4. Construct a strategy profile such that such that the players are alternating between (D, L) and (M, R) on the equilibrium path. How large δ needs to be so that the strategy profile you have just defined is a subgame perfect equilibrium?

Solution. 1. Pure Strategy Nash Equilibria: (U, R) and (D, M). Note that for player B, L is strictly dominated by M. In a NE, player B will never play L with a positive probability. Assume player B's strategy in the NE is $(0, p, 1-p)$. Player A's expected payoff of playing each strategy is then

$$EU_A(U) = 10(1-p)$$

$$EU_A(M) = 2p + 7(1-p)$$

$$EU_A(D) = 5p$$

i. if $0 < p < \frac{3}{5}$, $EU_A(U) > EU_A(M) > EU_A(D)$. Player A will only play U.

ii. if $p = \frac{3}{5}$, $EU_A(U) = EU_A(M) > EU_A(D)$. Player A will mix between U and M. Note that when D is eliminated, player B will choose R.

iii. if $\frac{3}{5} < p < \frac{7}{10}$, $EU_A(M) > \max\{EU_A(U), EU_A(D)\}$. Player A will choose to play M.

iv. if $p = \frac{7}{10}$, $EU_A(M) = EU_A(D) > EU_A(U)$.Player A will mix between M and D. Assume A's strategy is $(0, q, 1-q)$. According to Indifference Principle,

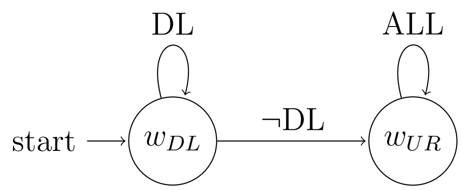
$$q + 10(1-q) = 5q \Rightarrow q = \frac{5}{7}$$

v. if $p > \frac{7}{10}$, $EU_A(D) > EU_A(M) > EU_A(U)$. Player A will choose to play D.

Therefore, there exists a unique mixed strategy NE $\left(\left(0, \frac{5}{7}, \frac{2}{7}\right), \left(0, \frac{7}{10}, \frac{3}{10}\right)\right)$.

2. Note that player B has no incentive to deviate from (M, R) but player A has incentive to deviate. So we should design a strategy profile such that player A will get punished in the second period in he deviates in the first period. In a finite repeated game, for a strategy profile to be a SPE, they must play a NE in the final stage. Therefore, we can construct a "stick and carrot" strategy: play (M, R) in the first stage; if player 1 deviates to (U, R) in the first stage, play (D, M) in the second stage; otherwise, play (U, R) in the second stage. We can prove that it is indeed a SPE by backward induction.

3. Consider the trigger strategy: initially play (D, L); if any deviation is observed, play (U, R) forever. The trigger strategy can be represented by the automaton:



$$\begin{cases} V_i(w_{DL}) = (1-\delta)\, u_i(DL) + \delta V_i(w_{DL}) \\ V_i(w_{UR}) = (1-\delta)\, u_i(UR) + \delta V_i(w_{UR}) \end{cases}$$

So we have $V_1(w_{DL}) = 8$, $V_1(w_{UR}) = 10$ and $V_2(w_{DL}) = 9$, $V_2(w_{UR}) = 4$. The normal form associated with w_{DL} is

	L	C	R
U	$(1-\delta)+10\delta,\ -(1-\delta)+4\delta$	$10\delta,\ 4\delta$	$10,\ 4$
M	$10\delta,\ 4\delta$	$2(1-\delta)+10\delta,\ (1-\delta)+4\delta$	$7(1-\delta)+10\delta,\ 5(1-\delta)+4\delta$
D	$8,\ 9$	$5(1-\delta)+10\delta,\ 10(1-\delta)+4\delta$	$10\delta,\ 4\delta$

Note that $0 \le \delta \le 1$, so for DL to be a NE, we must have

$$\begin{cases} 8 \ge (1-\delta)+10\delta & \Rightarrow \delta \le \frac{7}{9} \\ 9 \ge 10(1-\delta)+4\delta & \Rightarrow \delta \ge \frac{1}{6} \end{cases}$$

The normal form associated with w_{UR} is

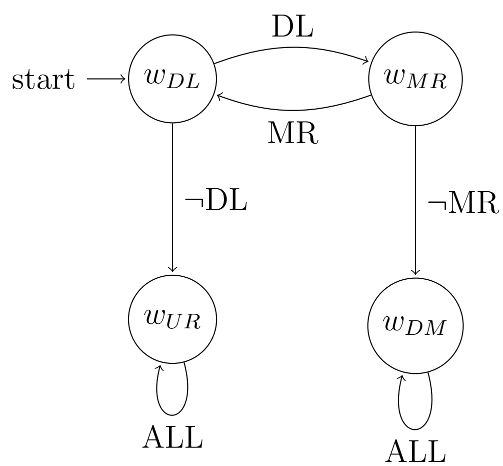
	L	C	R
U	$(1-\delta)+10\delta,\ -(1-\delta)+4\delta$	$10\delta,\ 4\delta$	$10,\ 4$
M	/	/	$7(1-\delta)+10\delta,\ 5(1-\delta)+4\delta$
D	/	/	$10\delta,\ 4\delta$

Incentive constraints at w_{UR} are:

$$\begin{cases} 10 \geq 7(1-\delta) + 10\delta & \Rightarrow \delta \leq 1 \\ 4 \geq -(1-\delta) + 4\delta & \Rightarrow \delta \leq 1 \end{cases}$$

Therefore, for the trigger strategy to be a SPE, we need to have $\frac{1}{6} \leq \delta \leq \frac{7}{9}$. Other potential strategies are left for you to have a brain storm.

4. Consider the modified trigger strategy: alternate between (D, L) and (M, R); if (D, L) is not realized when it should be, play (U, R) forever; similarly, if (M, R) is not realized when it should be, play (D, M) forever. This strategy can be represented by the automaton:



$$\begin{cases} V_i(w_{DL}) = (1-\delta)\, u_i(DL) + \delta V_i(w_{MR}) \\ V_i(w_{MR}) = (1-\delta)\, u_i(MR) + \delta V_i(w_{DL}) \\ V_i(w_{UR}) = (1-\delta)\, u_i(UR) + \delta V_i(w_{UR}) \\ V_i(w_{DM}) = (1-\delta)\, u_i(DM) + \delta V_i(w_{DM}) \end{cases}$$

It solves $V_1(w_{DL}) = \frac{8+7\delta}{1+\delta}$, $V_1(w_{MR}) = \frac{7+8\delta}{1+\delta}$, $V_1(w_{UR}) = 10$, $V_1(w_{DM}) = 5$ and $V_2(w_{DL}) = \frac{9+5\delta}{1+\delta}$, $V_1(w_{MR}) = \frac{5+9\delta}{1+\delta}$, $V_1(w_{UR}) = 4$, $V_1(w_{DM}) = 10$. (We skip the associated normal forms at each state.) The relevant incentive constraints are:

$$\begin{cases} \frac{8+7\delta}{1+\delta} \geq (1-\delta) + 10\delta & \Leftrightarrow 9\delta^2 + 3\delta - 7 \leq 0 \Rightarrow \delta \leq \frac{-3+\sqrt{261}}{18} \\ \frac{9+5\delta}{1+\delta} \geq 10(1-\delta) + 4\delta & \Leftrightarrow (2\delta+1)(3\delta-1) \geq 0 \Rightarrow \delta \geq \frac{1}{3} \\ \frac{7+8\delta}{1+\delta} \geq 10(1-\delta) + 5\delta & \Leftrightarrow 5\delta^2 + 3\delta - 3 \geq 0 \Rightarrow \delta \geq \frac{-3+\sqrt{69}}{10} \\ \frac{5+9\delta}{1+\delta} \geq (1-\delta) + 10\delta & \Leftrightarrow 9\delta^2 + \delta - 4 \leq 0 \Rightarrow \delta \leq \frac{-1+\sqrt{145}}{18} \end{cases}$$

Therefore, δ needs to be

$$\frac{-3+\sqrt{69}}{10} \leq \delta \leq \frac{-1+\sqrt{145}}{18}$$

In fact, we can state a similar strategy as: alternate between (D, L) and (M, R); only if player B deviates from (D, L), play (U, R) forever, but if player A deviates from (D, L), still go to (M, R); only if player A deviates from (M, R), play (D, M) forever, but if player B deviates from (M, R), still go to (D, L). In this way, player A has no incentive to deviate from (D, L) and player B has no incentive to deviate from (M, R). Then δ only needs to be $\delta \geq \frac{-3+\sqrt{69}}{10}$. In a word, the range of δ depends on how you define the strategy profile.

Exercise 3. Consider the following simultaneous move game.

	L	R
U	2, 2	x, 0
D	0, 5	1, 1

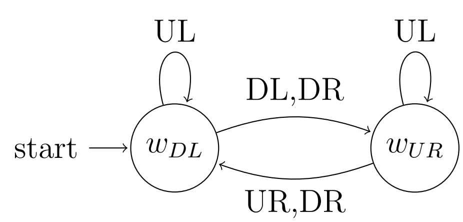
Let δ denote the common discount factor for both players and consider the strategy profile that induces the outcome path DL,UR,DL,UR, ..., and that, after any unilateral deviation by the row player specifies the outcome path DL,UR,DL,UR, ..., and after any unilateral deviation by the column player, specifies the outcome path UR,DL,UR,DL, ... (simultaneous deviations are ignored. i.e., are treated as if neither player had deviated).

1. Suppose $x = 5$. For what values of δ is this strategy profile subgame perfect?

2. Suppose now $x = 4$. How does this change your answer? Compare your answer with (1) and explain.

3. Suppose $x = 5$ again. Assume that at the beginning of each period, a coin is tossed. Each player can observe whether it is head or tail, and then takes the action. With probability $\frac{1}{2}$, it is head and DL is played; with probability $\frac{1}{2}$, it is a tail and UR is played. If any player deviates, then the outcome path is the same as the one specified at the beginning of the question. For what values of δ is this strategy profile subgame perfect? Compare your answer with (1) and explain.

Solution. Present the strategy by the automaton below



$$\begin{cases} V_i(w_{DL}) = (1-\delta)\,u_i(DL) + \delta V_i(w_{UR}) \\ V_i(w_{UR}) = (1-\delta)\,u_i(UR) + \delta V_i(w_{DL}) \end{cases}$$

It solves $V_1(w_{DL}) = \frac{x\delta}{1+\delta}$, $V_1(w_{UR}) = \frac{x}{1+\delta}$ and $V_2(w_{DL}) = \frac{5}{1+\delta}$, $V_2(w_{UR}) = \frac{5\delta}{1+\delta}$. The normal form associated with w_{DL} is

	L	R
U	$2(1-\delta)+\delta V_1(w_{DL}),\ 2(1-\delta)+\delta V_2(w_{DL})$	/
D	$\delta V_1(w_{UR}),\ 5(1-\delta)+\delta V_2(w_{UR})$	$(1-\delta)+\delta V_1(w_{UR}),\ (1-\delta)+\delta V_2(w_{UR})$

Note that $0 \le \delta \le 1$, so for DL to be a NE, we must have

$$\delta V_1(w_{UR}) \ge 2(1-\delta) + \delta V_1(w_{DL})$$

The normal form associated with w_{UR} is

	L	R
U	$2(1-\delta)+\delta V_1(w_{UR}),\ 2(1-\delta)+\delta V_2(w_{UR})$	$x(1-\delta)+\delta V_1(w_{DL}),\ \delta V_2(w_{DL})$
D	/	$(1-\delta)+\delta V_1(w_{DL}),\ (1-\delta)+\delta V_2(w_{DL})$

For UR to be a NE, we must have

$$\delta V_2(w_{DL}) \ge 2(1-\delta) + \delta V_2(w_{UR})$$

The non-deviation conditions are

$$\begin{cases} \delta\frac{x}{1+\delta} \ge 2(1-\delta) + \delta\frac{x\delta}{1+\delta} \\ \delta\frac{5}{1+\delta} \ge 2(1-\delta) + \delta\frac{5\delta}{1+\delta} \end{cases}$$

1. Suppose $x = 5$. Then we have $\delta\frac{5}{1+\delta} \ge 2(1-\delta) + \delta\frac{5\delta}{1+\delta}$, i.e., $(3\delta-2)(\delta-1) \le 0$, so $\delta \ge \frac{2}{3}$.

2. Suppose $x = 4$. Then an additional condition is $\delta\frac{4}{1+\delta} \ge 2(1-\delta) + \delta\frac{4\delta}{1+\delta}$, i.e., $(\delta-1)^2 \le 0$, so $\delta = 1$. When x decreases from 5 to 4, player A earns less from not deviating at the state DL. That is, player A has a stronger incentive to deviate to U if $x = 4$. To avoid deviation, we need a larger discount factor to guarantee that player A cares more about future payoffs.

3. The non-deviation conditions are

$$\begin{cases} 0 + \frac{5}{2}\delta \geq 2(1-\delta) + \delta V_1(w_{DL}) \\ 0 + \frac{5}{2}\delta \geq 2(1-\delta) + \delta V_2(w_{UR}) \end{cases}$$

Since $V_1(w_{DL}) = V_2(w_{UR}) = \frac{5\delta}{1+\delta}$, so we have

$$\frac{5}{2}\delta \geq 2(1-\delta) + \frac{5\delta^2}{1+\delta} \Leftrightarrow (\delta - 1)(\delta - 4) \leq 0 \Rightarrow \delta \geq 1$$

But $\delta \in (0,1)$.

4 Folk Theorem

From the above exercises, we can see that there can be many SPEs in infinitely repeated games. Can we always guarantee the existence of cooperation?

Theorem 2. *The Folk Theorem (Friedman 1971)*

When the discount factor is sufficiently large, any individual rational payoff vector can be supported in a subgame perfect equilibrium in an infinitely repeated game.

Game Theory Recitation 3: Incomplete Information

Xincheng Qiu*

Peking University

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Abstract

Common knowledge is sometimes too strong an assumption. Instead, it is often the case that one player knows something (i.e., his private information) while others don't, and this private information will affect the outcome of the game. Harsanyi introduced Bayesian games with "types" to model such incomplete information, and "type" enters into the payoff functions.

1 Static Bayesian Games

1.1 Set Up

- *n players*: $i = 1, 2, \ldots, n$.
- Nature chooses a *type* profile $t = (t_1, \ldots, t_n)$ where $t_i \in T_i$. Each player i is informed of his own type t_i but not others' types t_{-i}.
- However, each player i has *belief* about the distribution of others' types $p(t_{-i})$ (discrete probability) or $f(t_{-i})$ (continuous density).
- Players simultaneously choose *actions*. Note the difference between actions $a_i \in A_i$ and strategies $s_i \in S_i$. A player's (pure) *strategy* is a function $s_i(t_i)$ from types to actions, i.e., $s_i : T_i \to A_i$.
- *Payoffs* depend on both strategies and types $u_i(s_1, s_2, \ldots, s_n; t_1, t_2, \ldots, t_n)$.

*If you notice any typo, please drop me a line at xincheng.qiu@gmail.com. Comments are also greatly appreciated. The questions are designed for Problem Set 3 by Prof. Xi Weng at Guanghua School of Management, Peking University.

Note 1. *Imperfect Information v.s. Incomplete Information.*

- Imperfect Information: There exists some information sets that are not singletons. Put it differently, some player does not know which choices were made previously by other players when it is his turn to move.

- Incomplete Information: Player i's type t_i describe some information that is not common knowledge and only player i knows. In other words, some player does not know exactly which game he is playing (it depends on types).

1.2 Solution Concept: Bayesian Nash Equilibrium

Definition 1. The strategy profile $(s_1^*(t_1), \ldots, s_n^*(t_n))$ forms a *Bayesian Nash equilibrium* if for each player i and each of i's type $t_i \in T_i$, and for $\forall s_i \in S_i$

$$E_{t_{-i}}\left[u_i\left(s_i^*(t_i), s_{-i}^*(t_{-i}), t_i, t_{-i}\right) \mid t_i\right] \geq E_{t_{-i}}\left[u_i\left(s_i, s_{-i}^*(t_{-i}), t_i, t_{-i}\right) \mid t_i\right]$$

where the expectation over t_{-i} is taken with respect to the subjective belief $p(t_i)$. That is, $s_i^*(t_i)$ solves

$$\max_{s_i \in S_i} E_{t_{-i}}\left[u_i\left(s_i, s_{-i}^*(t_{-i}), t_i, t_{-i}\right) \mid t_i\right]$$

Remark 1. In a Bayesian Nash Equilibrium, each player chooses the best response to maximize his expected payoff conditional on his private information and corresponding beliefs[1].

2 Applications

		Type: Discrete	Type: Continuous
Strategy	Discrete	Payoff Matrix	Cutoff Point
Strategy	Continuous	Mixed Strategies	Linear Strategies

2.1 Continuous Types and Continuous Strategies: Linear Strategies

Question 1. *Consider a double auction where a seller is selling an indivisible object to a buyer. Let $v_s \sim U[0,1]$ denote the seller's valuation of the object and $v_b \sim U[0,1]$ denote the buyer's valuation. v_s and v_b are independent. Seller and buyer simultaneously propose*

[1] In case you forget about expectations, please see the appendix.

prices $p_s \in [0,1]$ and $p_b \in [0,1]$. Trade occurs at price $\gamma p_b + (1-\gamma)\,p_s$ if $p_b \geq p_s$; otherwise no trade. $\gamma \in [0,1]$ is a fixed parameter.

1. *Solve the Bayesian Nash equilibrium in linear strategies.*
2. *Calculate the buyer's expected payoffs when $\gamma = 0$ and $\gamma = 1$. Which γ does the buyer prefer, $\gamma = 0$ or $\gamma = 1$? Explain your result.*
3. *Calculate the expected probability that trade occurs. Which γ yields the highest expected probability of trading? Explain your result.*

Solution. 1. Seller's strategy: $p_s : [0,1] \to [0,1]$; Buyer's strategy: $p_b : [0,1] \to [0,1]$. Consider linear strategy: $p_s = \alpha_s + \beta_s v_s$ and $p_b = \alpha_b + \beta_b v_b$. For the strategy profile (p_s^*, p_b^*) to be a BNE, p_s^* maximizes

$$E_{v_b}\left[U_s\left(p_s, p_b^*, v_s\right) \mid v_s\right] = \int_{\frac{p_s-\alpha_b}{\beta_b}}^{1} \left[\gamma\left(\alpha_b + \beta_b v_b\right) + (1-\gamma)\,p_s - v_s\right] \mathrm{d}v_b$$

According to Leibniz integral rule[2], the F.O.C. is

$$-\left[\gamma\left(\alpha_b + \beta_b \frac{p_s - \alpha_b}{\beta_b}\right) + (1-\gamma)\,p_s - v_s\right]\frac{1}{\beta_b} + \int_{\frac{p_s-\alpha_b}{\beta_b}}^{1} (1-\gamma)\,\mathrm{d}v_b = 0$$

which yields

$$p_s^* = \frac{1}{2-\gamma}\left[(1-\gamma)\left(\alpha_b + \beta_b\right) + v_s\right] \tag{1}$$

Similarly, p_b^* maximizes

$$E_{v_s}\left[U_b\left(p_s^*, p_b, v_b\right) \mid v_b\right] = \int_{0}^{\frac{p_b-\alpha_s}{\beta_s}} \left[v_b - \gamma p_b - (1-\gamma)\left(\alpha_s + \beta_s v_s\right)\right] \mathrm{d}v_s$$

According to Leibniz integral rule, the F.O.C. is

$$\left[v_b - \gamma p_b - (1-\gamma)\left(\alpha_s + \beta_s \frac{p_b - \alpha_s}{\beta_s}\right)\right]\frac{1}{\beta_s} + \int_{0}^{\frac{p_b-\alpha_s}{\beta_s}} -\gamma \mathrm{d}v_s = 0$$

which yields

[2] If you are not familiar with Leibniz integral rule, please refer to the appendix.

$$p_b^* = \frac{1}{1+\gamma}\left(\gamma\alpha_s + v_b\right) \tag{2}$$

Combining equation 1 and equation 2, we have

$$\begin{cases} \alpha_s = \frac{1}{2-\gamma}\left(1-\gamma\right)\left(\alpha_b+\beta_b\right) \\ \beta_s = \frac{1}{2-\gamma} \\ \alpha_b = \frac{1}{1+\gamma}\gamma\alpha_s \\ \beta_b = \frac{1}{1+\gamma} \end{cases} \Rightarrow \begin{cases} \alpha_s = \frac{1-\gamma}{2} \\ \beta_s = \frac{1}{2-\gamma} \\ \alpha_b = \frac{\gamma(1-\gamma)}{2(1+\gamma)} \\ \beta_b = \frac{1}{1+\gamma} \end{cases}$$

The BNE in linear strategies is $p_s^* = \frac{1-\gamma}{2} + \frac{1}{2-\gamma}v_s$ and $p_b^* = \frac{\gamma(1-\gamma)}{2(1+\gamma)} + \frac{1}{1+\gamma}v_b$.

2. When $\gamma = 0$, $p_s^* = \frac{1}{2} + \frac{1}{2}v_s$ and $p_b^* = v_b$.

$$\begin{aligned} E_{v_s}\left[U_b\left(p_s^*, p_b^*, v_b\right) \mid v_b\right] &= \int_0^{2v_b-1}\left[v_b - \left(\frac{1}{2}+\frac{1}{2}v_s\right)\right]\mathrm{d}v_s \\ &= \left(v_b - \frac{1}{2}\right)^2 \end{aligned}$$

Note that it holds only when $2v_b - 1 \geq 0$, i.e., $v_b \geq \frac{1}{2}$. Actually, if $v_b < \frac{1}{2}$, $p_b < p_s$ and the trade never occurs. So

$$E_{v_s}\left[U_b \mid v_b\right] = \begin{cases} \left(v_b - \frac{1}{2}\right)^2, & \frac{1}{2} \leq v_b \leq 1 \\ 0 & 0 \leq v_b < \frac{1}{2} \end{cases}$$

When $\gamma = 1$, $p_s^* = v_s$ and $p_b^* = \frac{1}{2}v_b$.

$$\begin{aligned} E_{v_s}\left[U_b\left(p_s^*, p_b^*, v_b\right) \mid v_b\right] &= \int_0^{\frac{1}{2}v_b}\left[v_b - \frac{1}{2}v_b\right]\mathrm{d}v_s \\ &= \frac{1}{4}v_b^2 \end{aligned}$$

The ex ante expected payoff when $\gamma = 0$ is $EU_b = \int_{\frac{1}{2}}^1 \left(v_b - \frac{1}{2}\right)^2 \mathrm{d}v_b = \frac{1}{24}$. The ex-ante expected payoff when $\gamma = 1$ is $EU_b = \int_0^1 \frac{1}{4}v_b^2 \mathrm{d}v_b = \frac{1}{12}$. So the buyer prefers $\gamma = 1$.

3. The expected probability that trade occurs is

$$\Pr(p_b \geq p_s) = \iint_{p_b \geq p_s} \mathrm{d}v_s \mathrm{d}v_b$$
$$= \frac{1}{8}(1+\gamma)(2-\gamma)$$

F.O.C. yields $\gamma^* = \frac{1}{2}$.

Question 2. *Consider a first-price sealed-bid auction of an object between two risk-neutral bidders. Each bidder i (for $i = 1, 2$) simultaneously submits a bid $b_i \geq 0$. The bidder who submits the highest bid receives the object and pays his bid; both bidders win with equal probability in case they submit the same bid. Before the auction takes place, each bidder i privately observes the realization of a random variable t_i that is drawn independently from a uniform distribution over the interval $[0, 1]$.*

1. *Suppose first that the valuation of the object to bidder i is equal to $t_i + 0.5$. Therefore, the payoff of bidder i is $t_i + 0.5 - b_i$ if $b_i > b_j$; is $\frac{1}{2}(t_i + 0.5 - b_i)$ if $b_i = b_j$; is 0 if $b_i < b_j$. Derive the symmetric linear Bayesian Nash equilibrium for this game (i.e., each bidder uses an equilibrium strategy of the form $b_i = \alpha t_i + \beta$).*

2. *Now suppose the actual valuation of the object to bidder i is equal to $t_i + t_j$ $(j \neq i)$. Derive the symmetric linear Bayesian Nash equilibrium for this game (i.e., each bidder uses an equilibrium strategy of the form $b_i = \alpha t_i + \beta$).*

3. *Compare your answers in 1 and 2. Interpret your results.*

Solution. 1. Consider the symmetric linear BNE $b_i^* = \alpha t_i + \beta$. b_i^* maximizes player i's conditional expected payoff

$$E_{t_j}\left[U_i\left(b_i, b_j^*, t_i, t_j\right) \mid t_i\right] = \int_0^{\frac{b_i - \beta}{\alpha}} (t_i + 0.5 - b_i)\,\mathrm{d}t_j$$
$$= (t_i + 0.5 - b_i)\frac{b_i - \beta}{\alpha}$$

F.O.C. implies that $b_i^* = \frac{1}{2}t + \frac{1}{2}\beta + \frac{1}{4}$. That is, $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$.

2. Now the payoff of bidder i becomes

$$U_i(b_i, b_j, t_i, t_j) = \begin{cases} t_i + t_j - b_i, & b_i > b_j \\ \frac{1}{2}(t_i + t_j - b_i), & b_i = b_j \\ 0, & b_i < b_j \end{cases}$$

Again consider the symmetric linear BNE $b_i^* = \alpha t_i + \beta$. b_i^* maximizes player i's conditional expected payoff

$$E_{t_j}\left[U_i\left(b_i, b_j^*, t_i, t_j\right) \mid t_i\right] = \int_0^{\frac{b_i-\beta}{\alpha}} \left(t_i + t_j - b_i\right) \mathrm{d}t_j$$

According to Leibniz integral rule, the F.O.C. is

$$\left(t_i + \frac{b_i - \beta}{\alpha} - b_i\right)\frac{1}{\alpha} - \int_0^{\frac{b_i-\beta}{\alpha}} \mathrm{d}t_j = 0$$

which implies

$$b_i^* = \frac{1}{2\alpha - 1}\left[\alpha t_i + (\alpha - 1)\beta\right]$$

Therefore, $\alpha = 1$ and $\beta = 0$.

3. Notice that $b_i^{(1)} = \frac{1}{2}t_i + \frac{1}{2}$ and $b_i^{(2)} = t_i$, i.e., $b_i^{(1)} > b_i^{(2)}$ for $\forall t \in [0, 1)$.

2.2 Discrete Types and Continuous Strategies: Mixed Strategies

Question 3. *Fully characterize the symmetric BNE in the seat-taking example when* $v_i = 1 - \epsilon$ *with probability* $\frac{1}{2}$, $= 1 + \epsilon$ *with probability* $\frac{1}{2}$.

Solution. The payoff function of the seat-taking game is

$$U_i\left(v_i, s_i, s_j\right) = \begin{cases} v_i - s_i & s_i > s_j \\ \frac{1}{2}\left(v_i - s_i\right) & s_i = s_j \\ -s_i & s_i < s_j \end{cases}$$

We can first prove that there does not exist a pure strategy BNE by contradiction. Consider the symmetric mixed strategy BNE $f_L(s_i)$ when observing $v_i = 1 - \epsilon$ and $f_H(s_i)$ when observing $v_i = 1 + \epsilon$. Denote the cumulative distribution functions by F_L and F_H, respectively. We assume they have support $[0, \underline{s}]$ and $[\underline{s}, \overline{s}]$. Without loss of generality, player 1 is indifferent among the support, i.e., those strategies he plays with a positive density.

1. If $v_1 = 1 - \epsilon$. By indifference principle, each $s \in [0, \underline{s}]$ generates the same expected payoff

$$E\left[U_1\left(v_1, s, f_2^*\right) \mid v_L\right] = \frac{1}{2}\left[-s + \int_0^s (1 - \epsilon)\,\mathrm{d}F_L\right] + \frac{1}{2}\left[-s\right]$$

which does not vary as s changes. So the derivative with respect to s should be zero. Then $f_L(s) = \frac{2}{1-\epsilon}$.

2. If $v_1 = 1 + \epsilon$. Similarly, each $s \in [\underline{s}, \overline{s}]$ generates the same expected payoff

$$E\left[U_1\left(v_1, s, f_2^*\right) \mid v_H\right] = \frac{1}{2}\left[-s + \int_{\underline{s}}^{s} (1+\epsilon)\, \mathrm{d}F_H\right] + \frac{1}{2}\left[1 + \epsilon - s\right]$$

which does not vary as s changes. So the derivative with respect to s should be zero. Then $f_H(s) = \frac{2}{1+\epsilon}$.

Note that $F_L(\underline{s}) = \int_0^{\underline{s}} f_L(s) = \frac{2}{1-\epsilon}\underline{s} = 1$, so $\underline{s} = \frac{1-\epsilon}{2}$. And $F_H(\overline{s}) = \int_{\underline{s}}^{\overline{s}} f_H(s) = \frac{2}{1+\epsilon}\left(\overline{s} - \underline{s}\right) = 1$, so $\overline{s} = 1$.

Therefore, the symmetric BNE is $f_1^* = f_2^* = (f_L^*, f_H^*)$, where

$$f_L^*(s) = \frac{2}{1-\epsilon}, \ s \in \left[0, \frac{1-\epsilon}{2}\right]$$

$$f_H^*(s) = \frac{2}{1+\epsilon}, \ s \in \left[\frac{1-\epsilon}{2}, 1\right]$$

2.3 Continuous Types and Discrete Strategies: Cutoff Point

Question 4. *Consider the game depicted in the following table with two players, 1 and 2. The players simultaneously make their decisions. The states of the world are given by $(x, y) \in (0,1)^2$. Player 1 is told the first coordinate x of the state of world, and Player 2 is told the second coordinate y of the state of world. Each player, after learning his type, knows his payoff function, but does not know the payoff function of the other player.*

		2: L	R
1	U	$x, 0$	$0, y$
	D	$0, 1$	$1, 0$

For the first two questions, suppose that the common prior is such that both x and y are uniformly distributed on the interval $(0, 1)$, and the draws of x and y are independent.

1. *Show that there does not exist a Bayesian Nash equilibrium in which both players, of each type, use a completely mixed action.*

2. *Construct a Bayesian Nash equilibrium in which both players use a cutoff strategy (i.e. Player 1 chooses U if $x \geq \bar{x}$ and Player 2 chooses R if $y \geq \bar{y}$).*

3. *Now assume that given the value z that is told to a player, that player believes that the value told to the other player is uniformly distributed over the interval $(0, z)$. In other words, Player 1 believes that $x > y$ and Player 2 believes $y > x$. Construct a Bayesian Nash equilibrium in which both players, of each type, use a completely mixed action. Briefly explain why the results in (1) and (3) are different.*

Solution. 1. We can prove it by contradiction. Assume that player 1 chooses U with probability $p_1(x)$ and D with probability $1 - p_1(x)$; player 2 chooses L with probability $p_2(y)$ and R with probability $1 - p_2(y)$. Given player 2's strategy,

$$E_y [U_1 (U) \mid x] = \int_0^1 x p_2 (y) \, \mathrm{d}y$$

$$E_y [U_1 (D) \mid x] = \int_0^1 [1 - p_2 (y)] \, \mathrm{d}y$$

For player 1 to play mixed strategy, $E_y [U_1 (U) \mid x] = E_y [U_1 (D) \mid x]$, which implies

$$\int_0^1 p_2 (y) \, \mathrm{d}y = \frac{1}{1+x} \tag{3}$$

Similarly, given player 1's strategy,

$$E_x [U_2 (L) \mid y] = \int_0^1 1 - p_1 (x) \, \mathrm{d}x$$

$$E_x [U_2 (R) \mid y] = \int_0^1 y p_1 (x) \, \mathrm{d}x$$

For player 2 to play mixed strategy, $E_x [U_2 (L) \mid y] = E_x [U_2 (R) \mid y]$, which implies

$$\int_0^1 p_1 (x) \, \mathrm{d}x = \frac{1}{1+y} \tag{4}$$

However, it is impossible for equation 3 and equation 4 to always hold.

2. Consider a profile of cutoff strategies

$$s_1 (x) = \begin{cases} U & x \geq \bar{x} \\ D & x < \bar{x} \end{cases}$$

$$s_2 (y) = \begin{cases} R & y \geq \bar{y} \\ L & y < \bar{y} \end{cases}$$

From the property of the cutoff point, we know that player 1 is indifferent between U and D when $x = \bar{x}$.

$$E_y\left[U_1(U) \mid x = \bar{x}\right] = \int_0^{\bar{y}} \bar{x}\mathrm{d}y = \bar{x}\bar{y}$$

$$E_y\left[U_1(D) \mid x = \bar{x}\right] = \int_{\bar{y}}^{1} 1\mathrm{d}y = 1 - \bar{y}$$

Therefore

$$\bar{x}\bar{y} = 1 - \bar{y} \tag{5}$$

similarly, player 2 is indifferent between L and R when $y = \bar{y}$.

$$E_x\left[U_2(L) \mid y = \bar{y}\right] = \int_0^{\bar{x}} 1\mathrm{d}x = \bar{x}$$

$$E_x\left[U_2(R) \mid y = \bar{y}\right] = \int_{\bar{x}}^{1} \bar{y}\mathrm{d}x = \bar{y}(1 - \bar{x})$$

Therefore

$$\bar{x} = \bar{y}(1 - \bar{x}) \tag{6}$$

Combing equation 5 and equation 6 solves

$$\begin{cases} \bar{x} = \sqrt{2} - 1 \\ \bar{y} = \frac{\sqrt{2}}{2} \end{cases}$$

3. As in (1), assume that player 1 chooses U with probability $p_1(x)$ and D with probability $1 - p_1(x)$; player 2 chooses L with probability $p_2(y)$ and R with probability $1 - p_2(y)$. For player 1 to play the mixed strategy,

$$E_y\left[U_1(U) \mid x\right] = \int_0^x x p_2(y) \frac{1}{x}\mathrm{d}y = E_y\left[U_1(D) \mid x\right] = \int_0^x \left[1 - p_2(y)\right] \frac{1}{x}\mathrm{d}y$$

So we have

$$\int_0^x p_2(y)\,\mathrm{d}y = \frac{x}{1+x} \tag{7}$$

Similarly, for player 2 to play the mixed strategy,

$$E_x\left[U_2\left(L\right)\mid y\right]=\int_0^y\left[1-p_1\left(x\right)\right]\frac{1}{y}\mathrm{d}x=E_x\left[U_2\left(R\right)\mid y\right]=\int_0^y yp_1\left(x\right)\frac{1}{y}\mathrm{d}x$$

So we have

$$\int_0^y p_1\left(x\right)\mathrm{d}x=\frac{y}{1+y} \tag{8}$$

Equation 7 and 8 can be rewritten as

$$\int_0^y p_2\left(t\right)\mathrm{d}t=\frac{y}{1+y}$$

$$\int_0^x p_1\left(t\right)\mathrm{d}t=\frac{x}{1+x}$$

Taking derivatives[3], we can solve for

$$\begin{cases} p_1\left(x\right)=\left(\frac{x}{1+x}\right)'=\frac{1}{(1+x)^2} \\ p_2\left(y\right)=\left(\frac{y}{1+y}\right)'=\frac{1}{(1+y)^2} \end{cases}$$

2.4 Discrete Types and Discrete Strategies: Payoff Matrix

Example 1. Player 1 knows which of the following two games is played and Player 2 knows only that each game is played with equal probabilities. We can model such incomplete information as player 1's type $t_1=1,2$. Player 1 knows exactly his own type but player 2 only knows $\Pr\left(t_1=1\right)=\Pr\left(t_1=2\right)=\frac{1}{2}$.

$t_1=1$	X	Y
a	2, 2	4, 0
b	0, 4	3, 3

$t_1=2$	X	Y
x	0, 0	0, 0
y	0, 0	1, 1

Player 1's strategy set: {ax, ay, bx, by} and Player 2's strategy set {X, Y}. From an ex ante point of view, we can present the above game in a normal form:

	X	Y
ax	1, 1	2, 0
ay	1, 1	$\frac{5}{2},\frac{1}{2}$
bx	0, 2	$\frac{3}{2},\frac{3}{2}$
by	0, 2	2, 2

In the interim, this game can be represented as:

[3] The second fundamental theorem of calculus.

$t_1 = 1$	X	Y
ax	2, 1	4, 0
ay	2, 1	4, $\frac{1}{2}$
bx	0, 2	3, $\frac{3}{2}$
by	0, 2	3, 2

$t_1 = 2$	X	Y
ax	0, 1	0, 0
ay	0, 1	1, $\frac{1}{2}$
bx	0, 2	0, $\frac{3}{2}$
by	0, 2	1, 2

It can be verified that (ax, X) and (ay, X) are two BNE.

A Maths Appendix

A.1 Expectation

The expectation of a discrete random variable: $E[X] = \sum_i x_i p_i$

The expectation of a continuous random variable:

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} x \mathrm{d}F(x)$$

where $f(x)$ is the probability density function and $F(x)$ is the cumulative distribution function. The expectation of a function of x, $U(x)$ is

$$E[U(x)] = \int_{-\infty}^{\infty} U(x) f(x)\,\mathrm{d}x$$

Note 2. Don't forget $f(x)$! You may ignore this since in most of the exercises above, we simply assume a uniform distribution on $[0,1]$ whose density $f(x) = 1$.

A.2 Leibniz Integral Rule

Theorem 1. *Leibniz integral rule.*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{a(x)}^{b(x)} f(x,t)\,\mathrm{d}t\right) = f(x, b(x)) \cdot \frac{\mathrm{d}}{\mathrm{d}x} b(x) - f(x, a(x)) \cdot \frac{\mathrm{d}}{\mathrm{d}x} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,\mathrm{d}t$$

A.3 The Second Fundamental Theorem of Calculus

Theorem 2. *The second fundamental theorem of calculus.*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{a}^{x} f(t)\,\mathrm{d}t\right) = f(x)$$

Game Theory Recitation 4: Dynamic Games of Incomplete Information

Xincheng Qiu*
Peking University

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1 Equilibrium Concepts

1.1 Review

Nash Equilibrium is the most fundamental solution concept. It requires

1. every player chooses the best response given his belief about what others will do;

2. every player's belief is consistent with others' strategies.

SPE, PBE are refinements of NE with additional "reasonable" requirements. What is a "reasonable" equilibrium concept in dynamic games with incomplete information? Two requirements are imposed for PBE:

1. *Sequential Rationality*: The choices specified by b should be optimal at every information set given the beliefs specified by μ.

2. *Bayesian updating*: The beliefs specified by μ should be consistent with the strategy profile b.

Definition 1. *Beliefs.*

A system of beliefs μ in a finite extensive form game is a specification of a probability distribution over the decision nodes in every information set.

*If you notice any typo, please drop me a line at xincheng.qiu@gmail.com. Comments are also greatly appreciated.

Definition 2. *Sequential Rationality.*

A behavior strategy $\hat{b}_i$ in a finite extensive form game is sequentially rational given a system of beliefs μ, if

$$E^{\mu,\left(\hat{b}_i,\hat{b}_{-i}\right)}\left[u_i \mid h\right] \geq E^{\mu,\left(b_i,\hat{b}_{-i}\right)}\left[u_i \mid h\right]$$

for every information set h and for all b_i. We say that $\hat{b}$ is sequentially rational if, for every Player i, $\hat{b}_i$ is sequentially rational.

Definition 3. *Bayesian updating (on the path of play).*

(b, μ) satisfies Bayesian updating if, for every information set h with $P^b(h) > 0$ and node $t \in h$,

$$\mu(t) = \frac{P^b(t)}{P^b(h)}$$

1.2 Weak PBE

Definition 4. *Weak Perfect Bayesian Equilibrium.*

A strategy profile b of a finite extensive form game is a weak perfect Bayesian equilibrium if there exists a system of beliefs μ such that

1. b is *sequentially rational* given μ.
2. The belief on the path of play is derived from *Bayes' rule.*

Remark 1.

- A weak PBE consists of a pair of *strategies* and *beliefs*, (b, μ).
- *Weak Perfect Bayesian Equilibrium = Sequential Rationality (at every information set)+ Bayesian updating (on the path of play)*
- The belief off the path of play can be defined arbitrarily as long as sequential rationality is satisfied.
- Sequential rationality implies that no player chooses strictly dominated actions at any information set.
- Every weak PBE is a NE. But it is possible that a weak PBE is not a SPE.

1.3 Further Refinements

There are two kinds of information sets:

- Reached information sets: information sets on the path of play that are reached with positive probability;
- Unreached information sets: information sets off the path of play that are reached with zero probability.

Bayesian updating applies to information sets on the path of play, but cannot apply to information sets off the path of play, since Bayes' rule won't work if the denominator is zero. So far we have imposed no restriction on beliefs at unreached information sets in the concept of weak PBE. Now we want to extend the requirement of Bayesian updating that would also reasonably apply to unreached information sets. How should a rational player revise his beliefs when receiving information that is extremely surprising (with zero probability)? Game theorists have tried to narrow down the set of equilibria by adding more restrictions on off-equilibrium path beliefs.

Definition 5. *Almost Perfect Bayesian Equilibrium*: modifying *Bayesian updating* as

For any information set h' and following information set h reached with positive probability from h' under (μ, b),

$$\mu(t) = \frac{P^{\mu,b}(t \mid h')}{P^{\mu,b}(h_{h'} \mid h')} \sum_{t' \in h_{h'}} \mu(t') \qquad \forall t \in h_{h'}$$

where $h_{h'} \coloneqq \{t \in h : \exists t' \in h', t' \prec t\}$

Theorem 1. *Every almost perfect Bayesian equilibrium is subgame perfect.*

Definition 6. *Sequential Equilibrium.*

In a finite extensive form game, a system of beliefs μ is *consistent* with the strategy profile b if there exists a sequence of completely mixed sequence of behavior strategy profiles $\{b^k\}_k$ converging to b such that the associated sequence of system of beliefs $\{\mu^k\}_k$ obtained via Bayes' rule converges to μ.

A strategy profile b is a *sequential equilibrium* if it is sequentially rational at every information set, for some consistent system of beliefs μ.

Remark.

- *Sequential Equilibrium = Sequential Rationality + Consistent Belief.*

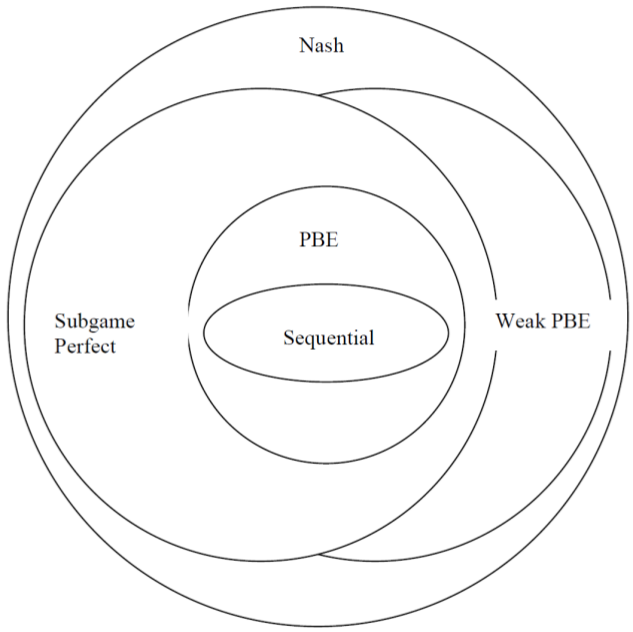

Figure 1: Summary

- Since consistency implies Bayesian updating at reached information sets, every sequential equilibrium is a weak PBE. The strategy profile also constitutes a SPE.
- (*Existence*) There exists at least one sequential equilibrium for every finite extensive form game (Kreps and Wilson, 1982).

Definition 7. *Intuitive Criterion (Cho and Krep, 1987)*: putting zero probability on the equilibrium-dominated thing.

2 Asymmetric Information

Definition 8. *Adverse Selection v.s. Moral Hazard.*

Adverse Selection: "hidden information" problems

Moral Hazard: "hidden action" problems

Definition 9. *Screening v.s. Signaling.*

Screening: the uninformed monopolist offers different contracts to separate out agents.

Signaling: an informed agent takes a costly action to signal to the uninformed monopolist.

Definition 10. *IR and IC.*

IR (*individual rationality*): agents prefer to sign the contract than not to.

IC (*incentive compatibility*): agents prefer to act according to his true type than to mimic others.

3 Problem Set[1]

Exercise 1. *Dynamic Pricing*

Consider a seller (Sheila) selling a nondurable good to a buyer (Bruce). There are two periods, and Bruce can purchase one unit in both, either, or neither periods. Both Bruce and Sheila discount by rate $\delta \in (0,1)$. Denote Bruce's reservation price by v, so that Bruce's payoff is given by $d_1(v-p_1)+\delta d_2(v-p_2)$, where p_t is the price charged by Sheila in period t, and $d_t = 1$ if Bruce purchases in period t and $= 0$ otherwise. Similarly, Sheila's payoff is $d_1 p_1 + \delta d_2 p_2$. It is common knowledge that Sheila has zero costs of production. There is incomplete information about Bruce's reservation price.

The game is as follows: In the first period, Sheila announces a price p_1. Nature then determines Bruce's reservation price (type) according to the probability distribution that assigns probability $\frac{1}{2}$ to $v = v_H$ and probability $\frac{1}{2}$ to $v = v_L$. Bruce learns his type and then decides whether to buy or not to buy. In the second period, Sheila again announces a price (knowing whether Bruce had bought or not in the first period) and Bruce then decides whether to buy or not. Assume $0 < 2v_L < v_H$. Restrict attention to pure strategies and assume that Bruce always buys when he is indifferent.

1. Describe Sheila's and Bruce's (extensive form) strategies of the two period game.

2. Consider a subgame starting at $p_1 \in (v_L, \delta v_L + (1-\delta)v_H]$. Describe a separating weak perfect Bayesian equilibrium of this subgame (different types of the buyer choose different actions in the first period) and verify this is indeed an equilibrium.

3. Consider a subgame starting at $p_1 \le v_L$. Describe a pooling weak perfect Bayesian equilibrium of this subgame and verify this is indeed an equilibrium.

4. Suppose Sheila must choose $p_1 \le \delta v_L + (1-\delta)v_H$ and the subsequent play starting at p_1 is specified by (2) and (3). Which price will Sheila charge in the first period?

Solution. 1. The timing of the game is specified as: S chooses $p_1 \to$ N chooses $v \to$ B chooses $d_1(v,p_1) \to$ S updates her belief and chooses $p_2(p_1,d_1) \to$ B chooses $d_2(v,p_1,d_1,p_2)$.

A strategy for Sheila is $(p_1, p_2(p_1,d_1))$, where $p_1 \in \mathbb{R}_+$ and $p_2(p_1,d_1) : \mathbb{R}_+ \times \{0,1\} \to \mathbb{R}_+$.

[1]The questions are designed for Problem Set 4 by Prof. Xi Weng at Guanghua School of Management, Peking University.

A strategy for Bruce is $(d_1(v,p_1), d_2(v,p_1,d_1,p_2))$,where $d_1(v,p_1) : \{v_H, v_L\} \times \mathbb{R}_+ \to \{0,1\}$, and $d_2(v,p_1,d_1,p_2) : \{v_H, v_L\} \times \mathbb{R}_+ \times \{0,1\} \times \mathbb{R}_+ \to \{0,1\}$.

2. Note that in a separating equilibrium, Sheila will assign probability 1 to v_H or v_L in the second period. She will charge a price of $p_2 = v_H$ or $p_2 = v_L$, respectively. Consider the following separating behavior:

$$d_1(v,p_1) = \begin{cases} 1, & v = v_H \\ 0, & v = v_L \end{cases}$$

$$p_2(p_1,d_1) = d_1 v_H + (1-d_1) v_L$$

$$d_2(v,p_1,d_1,p_2) = 1$$

and the belief

$$\mu(v_H \mid d_1 = 1) = 1$$

$$\mu(v_H \mid d_1 = 0) = 0$$

We need to check that both types have no incentive to deviate from $d_1(v,p_1)$. Since $p_1 > v_L$, the low type will not deviate to buy. For the high type, the above profile generates a payoff of $(v_H - p_1)$, while deviation generates $\delta(v_H - v_L)$. $(v_H - p_1) \geq \delta(v_H - v_L)$ guarantees that it is indeed a weak PBE.

3. Consider the following strategies

$$d_1(v,p_1) = 1,\ v = v_H \text{ and } v = v_L$$

$$p_2(p_1,d_1) = v_H$$

$$d_2(v,p_1,d_1,p_2) = \begin{cases} 1 & v = v_H \\ 0 & v = v_L \end{cases}$$

and the belief

$$\mu(v_H \mid d_1 = 1) = \frac{1}{2}$$

$$\mu(v_H \mid d_1 = 0) = \frac{1}{2}$$

4. If $v_L < p_1 \leq \delta v_L + (1-\delta) v_H$, Sheila's expected payoff is $\frac{1}{2}(p_1 + \delta v_H) + \frac{1}{2}(0 + \delta v_L)$, which increases with p_1. So the maximum payoff is

$$\pi_1 = \delta v_L + \frac{1}{2} v_H$$

If $p_1 \leq v_L$, Sheila's expected payoff is $p_1 + \frac{1}{2}\delta v_H$, which increases with p_1. So the maximum payoff is

$$\pi_2 = v_L + \frac{1}{2}\delta v_H$$

Note that $\pi_1 - \pi_2 = (1-\delta)\left(\frac{1}{2}v_H - v_L\right) \geq 0$, Sheila will charge $p_1^* = \delta v_L + (1-\delta) v_H$ in the first period.

Exercise 2. *Signaling Game*

Consider the following signaling game where player 1's type θ is either θ' (with probability p) or θ'' (with probability $1-p$).

Player 1 observes her type and chooses an action $a_1 \in \{U, D\}$. Player 2 observes 1's action but not 1's type, and chooses $a_2 \in \{L, R\}$.

When $\theta = \theta'$, the payoff matrix is:

	L	R
U	3, 3	0, 0
D	0, 0	2, 2

When $\theta = \theta''$, the payoff matrix is:

	L	R
U	1, -1	-1, 1
D	-1, 1	1, -1

1. Prove or disprove that the game have a "separating" perfect Bayesian equilibrium, where 1 takes different actions under different types. Specify the equilibrium if there exists a separating PBE. (Hint: Don't forget to specify 2's belief.)

2. Specify a pure strategy "pooling" perfect Bayesian equilibrium of this game. For what values of p will this be an equilibrium?

3. For what values of p does the game have a PBE in which type θ'' plays U with probability 1 and type θ' assigns strictly positive probability to both actions?

Solution. 1. Suppose player 1's strategy is playing U when observing θ' and playing D when observing θ''. By Bayesian updating, player 2's belief should be $\mu(\theta' \mid U) = 1$ and $\mu(\theta' \mid D) = 0$. Player 2's optimal choice is to play L when observing U and to play L also when observing D. But player 1 has incentive to deviate to U when observing θ'', since it yields a higher payoff $(-1 < 1)$. Thus it cannot constitute a PBE.

Suppose then player 1's strategy is playing D when observing θ' and playing U when observing θ''. By Bayesian updating, player 2's belief should be $\mu(\theta' \mid U) = 0$ and $\mu(\theta' \mid D) = 1$. Player 2's optimal choice is to play R when observing U and to play R also when observing D. But player 1 has incentive to deviate to D when observing θ'', since it yields a higher payoff $(-1 < 1)$. Thus it cannot constitute a PBE.

Therefore, there does not exist a separating PBE in this game.

2. Consider a pooling PBE in which player 1 chooses U no matter which type he is. Player 2's on-path belief should be $\mu(\theta' \mid U) = p$ and the off-path belief $\mu(\theta' \mid D) = q$ remains to be determined. $EU_2(L \mid U) = 3p - (1-p) = 4p - 1$ and $EU_2(R \mid U) = 0p + (1-p) = 1-p$. If $p \geq \frac{2}{5}$, L becomes 2's optimal choice given this belief. For player 1, U satisfies sequential rationality. For the information sets off the path, note that $EU_2(L \mid D) = 0q + (1-q) = 1-q$ and $EU_2(R \mid D) = 2q - (1-q) = 3q - 1$. If $q \leq \frac{1}{2}$, player 2 will choose L when (surprisingly) observing D. So the equilibrium can be described as

$$\left\{U, U; L, L; p \geq \frac{2}{5}, q \leq \frac{1}{2}\right\}$$

Similarly, we can also consider the pooling PBE candidate where player 1 always chooses D.

3. Suppose player 1's strategy is such PBE is $\sigma(\theta'')(U) = 1$ and $\sigma(\theta')(U) = x > 0$, where σ means the probability to play U. Player 2's belief is

$$\mu(\theta' \mid U) = \frac{P(U\theta')}{P(U\theta') + P(U\theta'')} = \frac{px}{px + 1 - p}$$

$$\mu(\theta' \mid D) = 1$$

Now let's check the sequential rationality requirement. So player 2's best response when observing D is R, i.e., $b_2(D) = R$. Note that when $b_2(D) = R$, type θ'' can play D and get the payoff 1. For U to be the optimal choice for type θ'', player 2's choice when observing U has to be L. But then type θ' will have no incentive to mix between U and D, since U will generate a higher payoff $(3 > 2)$. Therefore, this strategy profile cannot constitute a PBE.

Exercise 3. *Adverse Selection*

Consider a monopolistic market for cars. There are two consumers with quasi-linear preferences over cars and money. Consumer 1 is willing to buy at most one car and is willing to pay at most 7 for it. Consumer 2 is willing to buy at most two cars. He is willing to pay at most 10 for a single car and at most 15 for two cars. The monopolist meets one of these two consumers with equal probability. Cars are sold only in whole units and there are no costs.

1. Assume the monopolist can distinguish between the two consumers and can offer contracts that depend on the consumer's type. Find the optimal contracts.

2. Assume next that that the monopolist cannot observe the consumer's type. Find the optimal contract/s.

3. How would your answer to (2) change if consumer 1 was willing to pay at most 4 for a car (instead of 7)?

Solution. 1. If the monopolist can distinguish between the two consumers, the optimal contracts are $(q_1, p_1) = (1, 7)$ for consumer 1 and $(q_2, p_2) = (2, 15)$ for consumer 2.

2. Assume the monopolist cannot observe the consumer's type. Note that consumer 2 has higher willingness to pay than consumer 1, which implies that consumer 2 has an incentive to disguise as consumer 1 if the monopolist targets at consumer 1 only.

i) A single contract (q, p). As discussed above, we have $(q, p) = (2, 15)$, and the expected payoff for the monopolist is $\pi_1 = \frac{15}{2}$.

ii) Two contracts (q_1, p_1) and (q_2, p_2). The maximization problem for the monopolist is

$$\max \frac{1}{2}(p_1 + p_2)$$

$$\text{s.t.} \begin{cases} p_1 \leq 7 & IR_1 \\ p_2 \leq 15 & IR_2 \\ 15 - p_2 \geq 10 - p_1 & IC_2 \end{cases}$$

It solves $p_1^* = 7$ and $p_2^* = 12$. The expected payoff for the monopolist is $\pi_2 = \frac{1}{2}(7 + 12) = \frac{19}{2} > \pi_1 = \frac{15}{2}$.

The optimal contracts are $(q_1, p_1) = (1, 7)$ and $(q_2, p_2) = (2, 12)$.

3. If 7 is changed to 4, the two contracts case becomes

$$\max \frac{1}{2}(p_1 + p_2)$$

$$\text{s.t.}\begin{cases} p_1 \leq 4 & IR_1 \\ p_2 \leq 15 & IR_2 \\ 15 - p_2 \geq 10 - p_1 & IC_2 \end{cases}$$

It solves $p_1^* = 4$ and $p_2^* = 9$. The expected payoff for the monopolist is $\pi_2 = \frac{1}{2}(4+9) = \frac{13}{2} < \pi_1 = \frac{15}{2}$.

The optimal contract is $(q,p) = (2,15)$.

Exercise 4. *Regulating Natural Monopoly*

A public utility commission (the regulator) is charged with regulating a natural monopoly. The cost function of the natural monopoly is given by

$$C(q,\theta) = \begin{cases} 0, & q = 0 \\ K + \theta q, & q > 0 \end{cases}$$

where q is the quantity produced, $K > 0$ is the publicly known fixed cost, and $\theta \in (0,1)$ is marginal cost. The inverse demand curve for the good is $p(q) = \max\{1-2q, 0\}$. Supposing there are no income effects for this good, consumer surplus is given by

$$V(q) = \int_0^q p(\tilde{q})\, d\tilde{q} - p(q)\, q$$

The regulator determines the firm's regulated quantity q (with the regulated price given by $p(q)$) and subsidy s, as well as whether the firm is allowed to operate at all. The firm cannot be forced to operate.

The firm wishes to maximize expected profits,

$$\Pi(q) = p(q)\, q - C(q,\theta) + s$$

The regulator maximizes the total of consumer surplus and firm profits net of the subsidy,

$$V(q) + \Pi(q) - s = \int_0^q p(\tilde{q})\, d\tilde{q} - C(q,\theta)$$

1. Suppose the marginal cost $\theta > 0$ is publicly known. Solve the regulator's problem. Carefully analyze when it is optimal not to allow the firm to operate.

2. Suppose the regulator's beliefs assign probability $\alpha_i \in (0,1)$ to $\theta = \theta_i \in (0,1)$, where $\alpha_1 + \alpha_2 = 1$ and $\theta_1 < \theta_2$. Write down the regulator's optimization problem, being explicit about the IC and IR constraints.

3. Which IR constraint is implied by the other constraints, and why? One of the IC constraints is an implication of two other conditions holding with equality. Which one and why?

4. Suppose the regulator chooses $q_i > 0$ for both types of the firm. Solve the optimal q_1 and q_2. Compare your results in (1) and (4), and explain.

5. Suppose the regulator can choose to shut down one type of the firm. Which type will be shut down by the regulator? Resolve the regulator's optimization problem in this case.

6. From your results in (4) and (5), when is it optimal not to allow one type of the firm to operate? Explain the intuition.

Solution. 1. If the firm is allowed to operate, the regulator's problem is

$$\max_{q,s} V(q) + \Pi(q) - s = \int_0^q (1 - 2\tilde{q})\, d\tilde{q} - K - \theta q$$

$$\text{s.t. } \Pi(q) = (1 - 2q)\, q - K - \theta q + s \geq 0$$

It solves $q^* = \frac{1-\theta}{2}$ and $s^* \geq K$. The maximized social welfare $W^* = \left(\frac{1-\theta}{2}\right)^2 - K$. When $K > \left(\frac{1-\theta}{2}\right)^2$, it is optimal not to allow the firm to operate.

2. The regulator's optimization problem is

$$\max_{q_1,s_1,q_2,s_2} \sum_{i=1}^{2} \alpha_i \left[\int_0^{q_i} (1 - 2\tilde{q})\, d\tilde{q} - K - \theta_i q_i\right]$$

$$\text{s.t.} \begin{cases} (1 - 2q_1)\, q_1 - K - \theta_1 q_1 + s_1 \geq 0 & IR_1 \\ (1 - 2q_2)\, q_2 - K - \theta_2 q_2 + s_2 \geq 0 & IR_2 \\ (1 - 2q_1)\, q_1 - K - \theta_1 q_1 + s_1 \geq (1 - 2q_2)\, q_2 - K - \theta_1 q_2 + s_2 & IC_1 \\ (1 - 2q_2)\, q_2 - K - \theta_2 q_2 + s_2 \geq (1 - 2q_1)\, q_1 - K - \theta_2 q_1 + s_1 & IC_2 \end{cases}$$

3. IR constraint of type θ_1 is implied by the other constraints. If IR_2 and IC_1 are satisfied, IR_1 naturally holds. The reason is, $\theta_1 < \theta_2$ guarantees that type 1 earns a higher profit than type 2 when type 1 is mimicking type 2. IR_2 guarantees that type 2 earns a positive profit and IC_1 guarantees that type 1 earns a higher profit if he is not mimicking type 2.

IC constraint of type θ_2 is an implication of IC_1 holding with equality.

$$\begin{aligned}&[(1-2q_2)\,q_2-K-\theta_2 q_2+s_2]-[(1-2q_1)\,q_1-K-\theta_2 q_1+s_1]\\ =&[(1-2q_2)\,q_2-K-\theta_1 q_2+s_2]-[(1-2q_1)\,q_1-K-\theta_1 q_1+s_1]-\theta_2 q_2+\theta_1 q_2+\theta_2 q_1-\theta_1 q_1\\ =&(q_1-q_2)(\theta_2-\theta_1)\geq 0\end{aligned}$$

4. From discussion in (3), the regulator's optimization problem can be simplified as

$$\max_{q_1,s_1,q_2,s_2}\sum_{i=1}^{2}\alpha_i\left[\int_0^{q_i}(1-2\tilde{q})\,d\tilde{q}-K-\theta_i q_i\right]$$

$$\text{s.t.}\begin{cases}(1-2q_2)\,q_2-K-\theta_2 q_2+s_2\geq 0 & IR_2\\ (1-2q_1)\,q_1-K-\theta_1 q_1+s_1\geq(1-2q_2)\,q_2-K-\theta_1 q_2+s_2 & IC_1\\ (1-2q_2)\,q_2-K-\theta_2 q_2+s_2\geq(1-2q_1)\,q_1-K-\theta_2 q_1+s_1 & IC_2\end{cases}$$

It solves

$$\begin{cases}q_1^*=\frac{1-\theta_1}{2}\\ q_2^*=\frac{1-\theta_2}{2}\\ s_2^*\geq K\\ s_2^*+\frac{1-\theta_2}{2}(\theta_2-\theta_1)\leq s_1^*\leq s_2^*+\frac{1-\theta_1}{2}(\theta_2-\theta_1)\end{cases}$$

5. Type θ_2 with a higher cost will be shut down. Now the regulator's optimization problem becomes

$$\max_{q_1,s_1}\alpha_1\left[\int_0^{q_1}(1-2\tilde{q})\,d\tilde{q}-K-\theta_1 q_1\right]$$

$$\text{s.t.}\begin{cases}(1-2q_1)\,q_1-K-\theta_1 q_1+s_1\geq 0\\ (1-2q_1)\,q_1-K-\theta_2 q_1+s_1\leq 0\end{cases}$$

It solves $q_1^*=\frac{1-\theta_1}{2}$, $K\leq s_1^*\leq K+\frac{1-\theta_1}{2}(\theta_2-\theta_1)$.

6. In (4), the social welfare is

$$W_4=\alpha_1\left[\left(\frac{1-\theta_1}{2}\right)^2-K\right]+\alpha_2\left[\left(\frac{1-\theta_2}{2}\right)^2-K\right]$$

In (5), the social welfare is

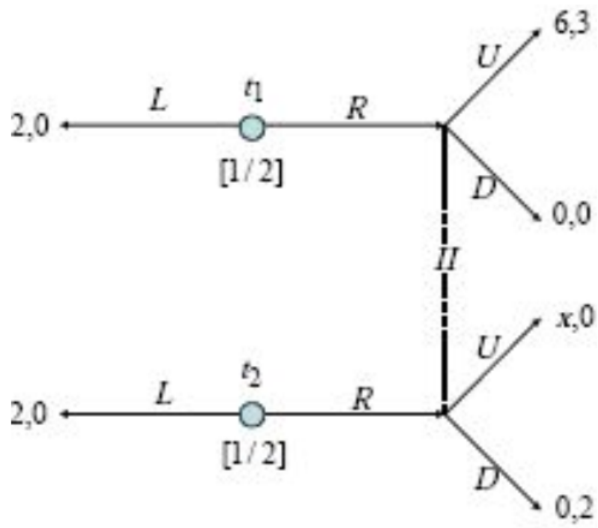

Figure 2: Game for Exercise 5

$$W_5 = \alpha_1 \left[\left(\frac{1-\theta_1}{2} \right)^2 - K \right]$$

If $K \geq \left(\frac{1-\theta_2}{2}\right)^2$, it optimal not to allow type θ_2 to operate.

Exercise 5. *Sequential Equilibrium v.s. Intuitive Criterion*

In the game illustrated by Figure 2, the probability that player I is type t_1 is $\frac{1}{2}$ and the probability that he is type t_2 is $\frac{1}{2}$. The first payoff is player I's payoff, and the second is player II's.

1. Describe a pooling weak perfect Bayesian equilibrium in which both types of player I play L, and verify this is indeed an equilibrium.

2. Show that, for all values of x, the outcome in which both types of player I play L is sequential by explicitly describing the converging sequence of completely mixed behavior strategy profiles and the associated system of beliefs.

3. For what values of x does this equilibrium pass the intuitive criterion?

Solution. 1. Consider the strategies

$$s_1(t_1) = s_1(t_2) = L$$

$$s_2(R) = D$$

and the beliefs

$$\mu(t_1 \mid L) = \mu(t_2 \mid L) = \frac{1}{2}$$

$$\mu(t_1 \mid R) \leq \frac{2}{5}$$

2. Suppose $\sigma_1(t_1) = 1 - x_n$, $\sigma_1(t_2) = 1 - y_n$ with $x_n \to 0$ and $y_n \to 0$. Since weak PBE only needs the off-path belief to be $\mu(t_1 \mid R) \leq \frac{2}{5}$, let's just take $\mu(t_1 \mid R) = \frac{1}{5}$. By Bayes' rule,

$$\mu(t_1 \mid L) = \frac{\frac{1}{2}(1 - x_n)}{\frac{1}{2}(1 - x_n) + \frac{1}{2}(1 - y_n)} = \frac{1 - x_n}{1 - x_n + 1 - y_n} \to \frac{1}{2}$$

which naturally holds. And

$$\mu(t_1 \mid R) = \frac{\frac{1}{2}x_n}{\frac{1}{2}x_n + \frac{1}{2}y_n} = \frac{x_n}{x_n + y_n} \to \frac{1}{5}$$

One example is $x_n = \frac{1}{n}$ and $y_n = \frac{4}{n}$.

3. If $x < 2$, player 2 believes t_2 type has no incentive to deviate to R and only t_1 type would possibly deviate to R. So player 2 will assign $\mu(t_2 \mid R) = 0$ and $\mu(t_1 \mid R) = 1$, which is contradictory to $\mu(t_1 \mid R) \leq \frac{2}{5}$. So this equilibrium does not pass the intuitive criterion. If $x \geq 2$, this equilibrium passes the intuitive criterion.

Suggested Solutions for the First Midterm *

Xincheng Qiu†

Peking University

April 2017

Exercise 1. Suppose Country A constructs facilities for the development of nuclear weapons. Country B sends a spy ring to Country A to ascertain whether it is developing nuclear weapons, and is considering bombing the new facilities. The spy ring sent by Country B is of quality $\alpha \in \left(\frac{1}{2}, 1\right)$: if Country A is developing nuclear weapons, Country B's spy ring will correctly report this with probability α, and with probability $1-\alpha$ it will report a false negative. If Country A is not developing nuclear weapons, Country B's spy ring will correctly report this with probability α, and with probability $1-\alpha$ it will report a false positive. Country A must decide whether or not to develop nuclear weapons, and Country B, after receiving its spy reports, must decide whether or not to bomb Country A's new facilities. The payoffs to the two countries appear in the following table.

		Country B: Bomb	Country B: Don't Bomb
Country A	Don't Develop	$\frac{1}{2}, \frac{1}{2}$	$\frac{3}{4}, 1$
	Develop	$0, \frac{3}{4}$	$1, 0$

1. Depict this situation as a normal-form game (i.e., write down the payoff matrix). For each $\alpha \in \left(\frac{1}{2}, 1\right)$, are there any strictly dominated strategies in the game?

2. For each $\alpha \in \left(\frac{1}{2}, 1\right)$, find the game's set of Nash equilibria.

3. Assuming both countries play their equilibrium strategy, what is the probability that Country A will manage to develop nuclear weapons without being bombed? How is this probability changing with α?

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*This problem set is designed by Prof. Xi Weng at Guanghua School of Management, Peking University.

†If you notice any errors or have any comments, please drop me a line at xincheng.qiu@gmail.com

Solution. First describe the strategy space for each player. $S_A = \{N, D\}$, where N stands for "Don't Develop" and D is short for "Develop". $S_B = \{BB, BN, NB, NN\}$, where BB means that country B will "Bomb" if the spy reports a nuclear weapon and will still "Bomb" if the spy reports no nuclear weapons. Similarly, BN represents a strategy that country B will "Bomb" if the spy reports a nuclear weapon and will "Not Bomb" if the spy reports no nuclear weapons. NB and NN are defined in the same way.

1. The payoff matrix is in essence a concise representation of the payoff functions.

Country B

Country A		BB	BN	NB	NN
	N	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2} + \frac{\alpha}{4}, \frac{1}{2} + \frac{\alpha}{2}$	$\frac{3}{4} - \frac{\alpha}{4}, 1 - \frac{\alpha}{2}$	$\frac{3}{4}, 1$
	D	$0, \frac{3}{4}$	$1 - \alpha, \frac{3}{4}\alpha$	$\alpha, \frac{3}{4} - \frac{3}{4}\alpha$	$1, 0$

For Country A, there is no dominated strategy. For Country B, NB is strictly dominated by BN. Note that $\alpha \in \left(\frac{1}{2}, 1\right)$, so

$$\left(\frac{1}{2} + \frac{\alpha}{2}\right) - \left(1 - \frac{\alpha}{2}\right) = \alpha - \frac{1}{2} > 0$$

$$\frac{3}{4}\alpha - \left(\frac{3}{4} - \frac{3}{4}\alpha\right) = \frac{3}{4}(2\alpha - 1) > 0$$

2. Since NB is a strictly dominated strategy, Country B will never play NB with a positive probability in a Nash equilibrium. We can thus eliminate NB from the payoff matrix.

Country B

Country A		BB	BN	NN
	N	$\underline{\frac{1}{2}}, \frac{1}{2}$	$\underline{\frac{1}{2} + \frac{\alpha}{4}}, \frac{1}{2} + \frac{\alpha}{2}$	$\frac{3}{4}, \underline{1}$
	D	$0, \underline{\frac{3}{4}}$	$1 - \alpha, \frac{3}{4}\alpha$	$\underline{1}, 0$

There is no pure strategy equilibrium in this game. Consider a mixed strategy equilibrium, in which Country A plays N with probability p and D with probability $1 - p$. For Country B, the expected payoff of each strategy now becomes:

$$\mathbb{E}\mathrm{U}_{\mathrm{B}}(BB) = \frac{1}{2}p + \frac{3}{4}(1 - p) = -\frac{1}{4}p + \frac{3}{4}$$

$$\mathbb{E}\mathrm{U}_{\mathrm{B}}(BN) = \left(\frac{1}{2} + \frac{\alpha}{2}\right)p + \frac{3}{4}\alpha(1 - p) = \left(\frac{1}{2} - \frac{1}{4}\alpha\right)p + \frac{3}{4}\alpha$$

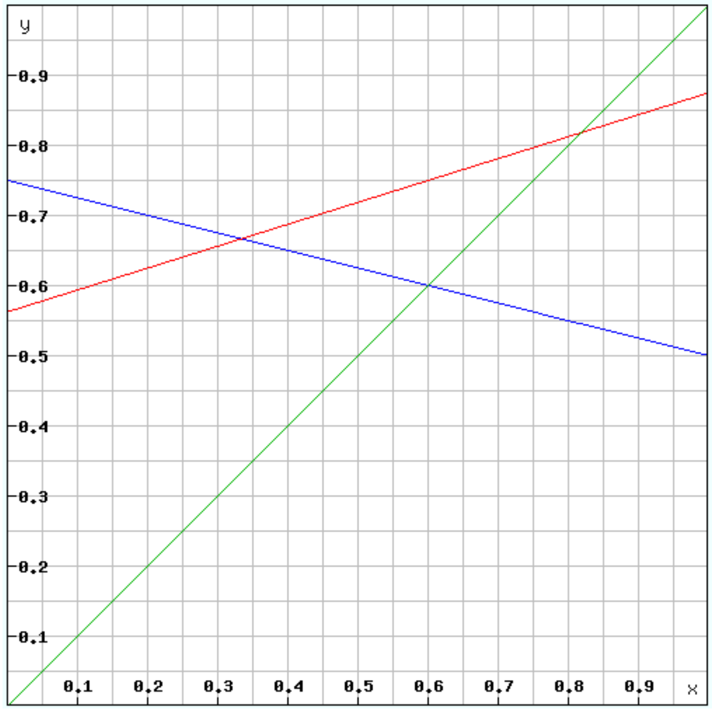


Figure 1: Country B's Expected Payoff of Each Strategy

$$\mathbb{E}\mathrm{U}_{\mathrm{B}}(NN) = p$$

Country B's expected payoff of each strategy is depicted in figure 1(where x axis is for p and y axis for the expected payoff; Blue for $\mathbb{E}\mathrm{U}_{\mathrm{B}}(BB)$, Red for $\mathbb{E}\mathrm{U}_{\mathrm{B}}(BN)$ and Green for $\mathbb{E}\mathrm{U}_{\mathrm{B}}(NN)$). $\mathbb{E}\mathrm{U}_{\mathrm{B}}(BB)(p)$ and $\mathbb{E}\mathrm{U}_{\mathrm{B}}(BN)(p)$ intersect at $p_1 = \frac{3-3\alpha}{3-\alpha} \in \left(0, \frac{3}{5}\right)$. $\mathbb{E}\mathrm{U}_{\mathrm{B}}(BB)(p)$ and $\mathbb{E}\mathrm{U}_{\mathrm{B}}(NN)(p)$ intersect at $p_2 = \frac{3}{5}$. $\mathbb{E}\mathrm{U}_{\mathrm{B}}(BN)(p)$ and $\mathbb{E}\mathrm{U}_{\mathrm{B}}(NN)(p)$ intersect at $p_3 = \frac{3\alpha}{2+\alpha} \in \left(\frac{3}{5}, 1\right)$. Now it is clear from Figure 1 that:

(a) when $0 \leq p < \frac{3-3\alpha}{3-\alpha}$, Country B's best response is to play BB;

(b) when $p = \frac{3-3\alpha}{3-\alpha}$, Country B is indifferent between BB and BN;

(c) when $\frac{3-3\alpha}{3-\alpha} < p < \frac{3\alpha}{2+\alpha}$, Country B's best response is to play BN;

(d) when $p = \frac{3\alpha}{2+\alpha}$, Country B is indifferent between BN and NN;

(e) when $\frac{3\alpha}{2+\alpha} < p \leq 1$, Country B's best response is to play NN.

Therefore, for Country B to play mixed strategies, $p = \frac{3-3\alpha}{3-\alpha}$ or $p = \frac{3\alpha}{2+\alpha}$.[1]

[1] I will only deduct 1 point for arithmetic mistakes in the calculation, as long as you were clear about the way of looking for the mixed strategy Nash equilibrium.

Case 1. $p = \frac{3-3\alpha}{3-\alpha}$. Assume that Country B plays BB with probability q_1 and BN with probability $1-q_1$. For Country A, the expected payoff of each strategy now becomes:

$$\mathbb{E}\mathrm{U}_{\mathrm{A}}(N) = \frac{1}{2}q_1 + \left(\frac{1}{2} + \frac{\alpha}{4}\right)(1-q_1)$$

$$\mathbb{E}\mathrm{U}_{\mathrm{A}}(D) = (1-\alpha)(1-q_1)$$

Indifference Principle implies that $\mathbb{E}\mathrm{U}_{\mathrm{A}}(N) = \mathbb{E}\mathrm{U}_{\mathrm{A}}(D)$, i.e., $q_1 = \frac{2-5\alpha}{4-5\alpha} \in \left(-\infty, -\frac{1}{3}\right) \cup (3, +\infty)$, which could not be a mixed strategy.

Case 2. $p = \frac{3\alpha}{2+\alpha}$. Assume that Country B plays BN with probability q_2 and NN with probability $1-q_2$. For Country A, the expected payoff of each strategy now becomes:

$$\mathbb{E}\mathrm{U}_{\mathrm{A}}(N) = \left(\frac{1}{2} + \frac{\alpha}{4}\right)q_2 + \frac{3}{4}(1-q_2)$$

$$\mathbb{E}\mathrm{U}_{\mathrm{A}}(D) = (1-\alpha)q_2 + (1-q_2)$$

Indifference Principle implies that $\mathbb{E}\mathrm{U}_{\mathrm{A}}(N) = \mathbb{E}\mathrm{U}_{\mathrm{A}}(D)$, i.e., $q_2 = \frac{1}{5\alpha-1} \in \left(\frac{1}{4}, \frac{2}{3}\right)$.

Therefore, the unique Nash equilibrium is the mixed strategy profile $\left\{\left(\frac{3\alpha}{2+\alpha}, \frac{2-2\alpha}{2+\alpha}\right), \left(0, \frac{1}{5\alpha-1}, 0, \frac{5\alpha-2}{5\alpha-1}\right)\right\}$.

3. The probability that Country A will manage to develop nuclear weapons without being bombed is

$$\Pr(\text{develop w/o bombed}) = \frac{2-2\alpha}{2+\alpha} \times \left[\frac{1}{5\alpha-1} \times (1-\alpha) + \frac{5\alpha-2}{5\alpha-1}\right] = \frac{(4\alpha-1)(2-2\alpha)}{(5\alpha-1)(2+\alpha)}$$

$$\begin{aligned}\frac{\partial \Pr}{\partial \alpha} &= \frac{(-16\alpha+10)(5\alpha^2+9\alpha-2) - (-8\alpha^2+10\alpha-2)(10\alpha+9)}{(5\alpha-1)^2(2+\alpha)^2} \\ &= \frac{-2(61\alpha^2-26\alpha+1)}{(5\alpha-1)^2(2+\alpha)^2}\end{aligned}$$

The solutions of $61x^2 - 26x + 1 = 0$ are $x = \frac{13 \pm 6\sqrt{3}}{61}$. Note that $\frac{13+6\sqrt{3}}{61} < \frac{1}{2}$. When $\alpha \in \left(\frac{1}{2}, 1\right)$, $61\alpha^2 - 26\alpha + 1 > 0$, so $\frac{\partial \Pr}{\partial \alpha} < 0$. This probability decreases as α increases.

Exercise 2. Suppose that two people can invest in a project that would be valuable to both of them if it is completed. The project requires 2 units of investment and will give a value of 25 to each person if undertaken. Person 1 can only invest in odd periods while person 2 can only invest in even periods. There are four periods, with the cost of investment in period t equal to $2t$ per unit of investment. That is, an investment of 1 costs 2 in period 1, 4 in period 2, 6 in period 3 and 8 in period 4. If there are at least 2 units of investment at the end of period 4 each person receives a payoff of 25 minus the cost of the investments that player has made. If there are 0 or 1 units of investment at the end of period 4 each person just loses the cost of her investment, if any. Suppose that in each period that a player can make an investment, either 0 or 1 can be invested.

1. Write down the extensive form tree for the game. What is the subgame perfect equilibrium for the game? (Be sure to describe precisely the strategies in the equilibrium.)

2. Suppose now that in each period that a player can make an investment, each player can invest 0, 1, or 2. What is the subgame perfect equilibrium?

3. Suppose now that as before, in each period that a player can make an investment, that player can invest 0 or 1, but that there are six periods instead of four periods. Also as before, the cost of making an investment of 1 is $2t$, where t is the period the investment is made. Thus, the cost of investing 1 in period 5 in 10 and the cost in period 6 is 12. What is the subgame perfect equilibrium outcome in this case? (You do not need to specify fully the equilibrium strategies in this case, but you should explain the logic briefly. There is also no need to write out the extensive form tree for the game.)

Solution. (Dynamic Game)

1. The extensive form is shown in figure 2.

   The subgame perfect equilibrium is $\{(I, INNN), (NI, NIININNN)\}$.

2. Suppose that each player can invest 0, 1, or 2 in each period. In the 4th stage, player 2 will invest 2 if there is no investment (since $25 - 16 = 9 > 0$), will invest 1 if there is 1 investment (since $25 - 8 = 17 > 0$), and will not invest if there are at least 2 units of investment. Expecting player 2's behavior in the final stage, player 1 will never invest any unit in the 3rd stage, since player 2's choice will always guarantee that there are at least 2 units of investment. Again, player 2 anticipate that player

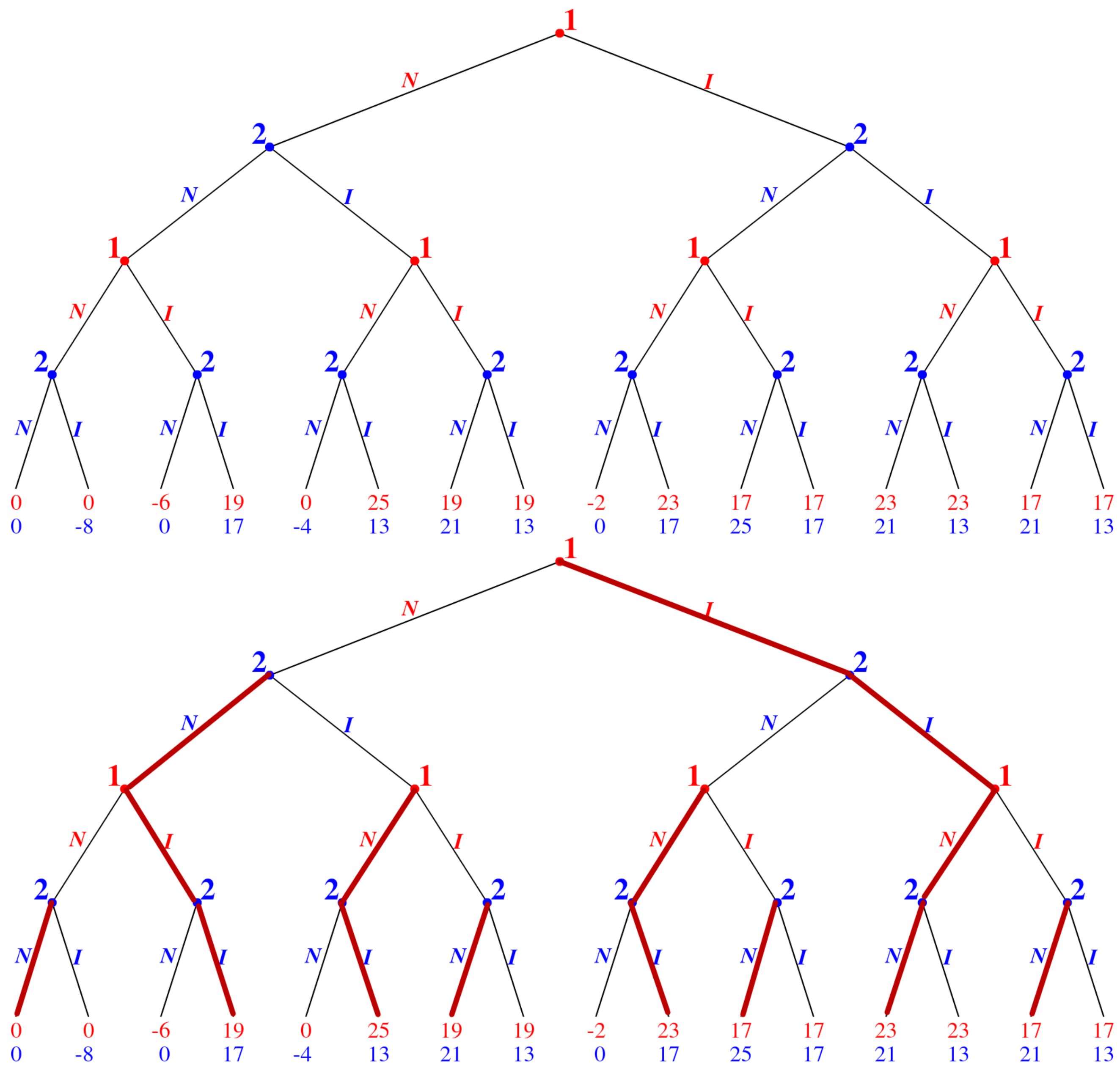


Figure 2: Extensive Form and Backward Induction

1 will not invest in the 3rd stage and that player 2 himself will fulfill the investment in the 4th stage, player 2 prefers to invest in the 2nd stage (at a lower cost). By the same logic, imagine that player 1 can foresee what player 2 will do in the 2nd stage, player 1 will not invest in the 1st stage. The subgame perfect equilibrium is thus $\{(0, 000000000), (210, 2101000001000000000000000000)\}$, yielding an outcome where player 1 will not invest in the 1st stage, player 2 will invest 2 units in the 2nd units, and there is no more investment in the following stages. The outcome payoff vector is$(25, 17)$.[2]

3. We perform backward induction. In the 6th stage, player 2 will invest if there is one unit of investment (since player 2's minimum payoff would be $25 - 12 - 8 = 5 > 0$), will not invest if there is no investment or if there are at least 2 units of investment. Player 1 hopes to see that project can be completed. Therefore, in the 5th stage, if there are already at least 2 units of investment, player 1 will not invest; if there are already one units of investment, player 1 will not invest either, since player 2 will invest in the final stage; if there is no investment, player 1 will invest one unit, anticipating that player 2 will continue to invest. In the 4th stage, player 2 will not invest if there are at least 2 units, will invest if there is only 1 unit (otherwise he will invest in the 6th stage at a higher cost, since player 1 will not invest in this case), and will not invest if there is no investment since player 1 will invest in the 5th stage. By the same logic, in the 3rd stage, player 1 will not invest if there are at least 1 unit and will invest if there is no investment. In the 2nd stage, player 2 will invest if there is 1 unit and will not invest otherwise. In the first stage, if player 1 invests, his payoff is 23; if he does not invest, his payoff is 19. Therefore player 1 prefers to invest in the 1st stage. The equilibrium outcome is that player 1 invests in the 1st stage and player 2 invests in the 2nd stage, and there is no further investment in the following stages, yields a payoff vector of $(23, 21)$.

Exercise 3. Consider a Hotelling duopoly model where consumers are uniformly distributed on the interval $[0, 1]$ and are of mass 1. A consumer of type x is located at some point x on the interval $[0, 1]$. Consumers buy up to one unit from one of the firms. Firm i is located at l_i somewhere on the interval $[0, 1]$, it charges price p_i, and consumers have to travel to the firm if they decide to visit it. The transportation cost is $t = \tau (x - l_i)^2$. If a consumer of type x buys product i, she then derives utility

[2] There are various possible notations for the strategy profile. The key is you are explicit in the notation about the idea that the strategy is a complete contingent plan for every situation.

$$v_i(x) = r - \tau (x - l_i)^2 - p_i$$

where r is each consumer's reservation value. The marginal cost of production is normalized to 0. In such a market, we analyze a duopoly in which firms first simultaneously decide which location to pick (or, equivalently, which product to produce) and secondly, they simultaneously set prices.

1. Suppose government regulation requires $p_1 = p_2 = p$ with $r - \tau - p \geq 0$. Find all Nash equilibria in pure strategies when the firms simultaneously choose the location.

2. Suppose government regulation is relaxed. The firms can freely choose both locations and prices. However, the location is restricted to be on the interval $[0, 1]$. We analyze a model in which firms first simultaneously decide which location to pick and secondly, after observing the locations, they simultaneously set prices. Given the locations l_1 and l_2, find the equilibrium prices set in the second stage.

3. Find the locations l_1 and l_2 chosen in the subgame perfect equilibrium. Explain why your result is different from the one in 1.

4. Suppose government regulation is further relaxed such that there is no restriction on the locations (firms can choose locations outside the interval $[0, 1]$). What are the locations l_1 and l_2 chosen in the subgame perfect equilibrium?

Solution. (Hotelling Duopoly Model)

1. The condition $r - \tau - p \geq 0$ guarantees that every consumer will buy one unit (from the nearest firm). We solve for the Nash equilibrium of the simultaneous location game. Firm 1's best response is

$$l_1^*(l_2) = \begin{cases} l_2 + \varepsilon & 0 \leq l_2 < \frac{1}{2} \\ l_2 & l_2 = \frac{1}{2} \\ l_2 - \varepsilon & \frac{1}{2} < l_2 \leq 1 \end{cases}$$

   where $\varepsilon > 0$ and $\varepsilon \to 0$. Similarly we can write down firm 2's best response. They intersect at the point $l_1 = l_2 = \frac{1}{2}$. This constitutes the unique pure strategy Nash equilibrium: both firms locate at the center.

   Remark 1. This is the standard Median Voter Theorem. The key insight is that although firms are able to differentiate products, it turns out that they will not. To grasp as much market share as possible, they locate at the center to cater to consumer tastes.

2. Note that if $l_1 = l_2$, this leads to the standard Bertrand competition model: $p_1^* = p_2^* = 0$. Now let us consider the case $l_1 \neq l_2$. Without loss of generality, suppose $l_1 < l_2$. The key is to find the consumer x^* such that she is indifferent between the two products.

$$r - \tau (x^* - l_1)^2 - p_1 = r - \tau (x^* - l_2)^2 - p_2$$

which implies that

$$x^* = \frac{p_2 - p_1}{2\tau (l_2 - l_1)} + \frac{l_1 + l_2}{2} \tag{1}$$

Consumers with $x < x^*$will buy from firm 1 and Consumers with $x > x^*$will go to firm 2. The profits of the two firms are

$$\pi_1 = p_1 x^* = \left(\frac{p_2 - p_1}{2\tau (l_2 - l_1)} + \frac{l_1 + l_2}{2} \right) p_1$$
$$\pi_2 = p_2 (1 - x^*) = \left(1 - \frac{p_2 - p_1}{2\tau (l_2 - l_1)} - \frac{l_1 + l_2}{\frac{1}{2}} \right) p_2$$

By F.O.C. we can derive the best responses of the two firms.

$$\frac{\partial \pi_1}{\partial p_1} = 0 \Rightarrow p_1^* (p_2) = \frac{1}{2} \left[\tau \left(l_2^2 - l_1^2 \right) + p_2 \right]$$
$$\frac{\partial \pi_2}{\partial p_2} = 0 \Rightarrow p_2^* (p_1) = \frac{1}{2} \left[2\tau (l_2 - l_1) - \tau \left(l_2^2 - l_1^2 \right) + p_1 \right]$$

From these two equations, we can solve for the equilibrium prices

$$\begin{cases} p_1^* (l_1, l_2) = \frac{\tau}{3} (l_2 - l_1) [2 + l_1 + l_2] \\ p_2^* (l_1, l_2) = \frac{\tau}{3} (l_2 - l_1) [4 - l_1 - l_2] \end{cases} \tag{2}$$

Note that if $0 \leq l_1 < l_2 \leq 1$, both $p_1^* (l_1, l_2)$ and $p_2^* (l_1, l_2)$ are positive.

3. After simple algebra using Equation 1 and 2, firms' problems can be written as

$$\max_{l_1 \in [0,1]} \pi_1 = p_1^* x^* = \frac{\tau}{18} (l_2 - l_1) (l_1 + l_2 + 2)^2$$

$$\max_{l_2 \in [0,1]} \pi_2 = p_2^* (1 - x^*) = \frac{\tau}{18} (l_2 - l_1) (4 - l_1 - l_2)^2$$

Again we can derive the best responses of the two firms by F.O.C.

$$\begin{cases} \frac{\partial \pi_1}{\partial l_1} = \frac{\tau}{18} (l_1 + l_2 + 2) (l_2 - 3l_1 - 2) \\ \frac{\partial \pi_2}{\partial l_2} = \frac{\tau}{18} (4 - l_1 - l_2) (l_1 - 3l_2 + 4) \end{cases} \tag{3}$$

Note that when locations are restricted to be on the interval $[0, 1]$, $l_1 + l_2 + 2 > 0$ and $4 - l_1 - l_2 > 0$, so the first order derivatives have the same signs as $l_2 - 3l_1 - 2$ and $l_1 - 3l_2 + 4$, respectively. Combine the two equations we solve for the equilibrium

$$\begin{cases} l_1^* = -\frac{1}{4} \\ l_2^* = \frac{5}{4} \end{cases}$$

However, the location is restricted to be on the interval $[0, 1]$. The equilibrium location decisions should be $l_1^* = 0$ and $l_2^* = 1$. (Note that $\frac{\partial \pi_1}{\partial l_1}\Big|_{l_1=0,l_2=1} < 0$ and $\frac{\partial \pi_2}{\partial l_2}\Big|_{l_1=0,l_2=1} > 0$)

4. If there is no restriction on the locations, consider again the F.O.C.s in Equation 3. There are four potential cases $\begin{cases} l_1 + l_2 + 2 = 0 \\ 4 - l_1 - l_2 = 0 \end{cases}$, $\begin{cases} l_1 + l_2 + 2 = 0 \\ l_1 - 3l_2 + 4 = 0 \end{cases}$, $\begin{cases} l_2 - 3l_1 - 2 = 0 \\ 4 - l_1 - l_2 = 0 \end{cases}$, $\begin{cases} l_2 - 3l_1 - 2 = 0 \\ l_1 - 3l_2 + 4 = 0 \end{cases}$, with corresponding solutions $\emptyset$, $\begin{cases} l_1^* = -\frac{5}{2} \\ l_2^* = \frac{1}{2} \end{cases}$, $\begin{cases} l_1^* = \frac{1}{2} \\ l_2^* = \frac{7}{2} \end{cases}$, $\begin{cases} l_1^* = -\frac{1}{4} \\ l_2^* = \frac{5}{4} \end{cases}$, respectively. Remember to verify the S.O.C.s

$$\begin{cases} \frac{\partial^2 \pi_1}{\partial l_1^2} = \frac{\tau}{18} (-6l_1 - 2l_2 - 8) \\ \frac{\partial^2 \pi_2}{\partial l_2^2} = \frac{\tau}{18} (2l_1 + 6l_2 - 16) \end{cases}$$

The only possible solution is $\begin{cases} l_1^* = -\frac{1}{4} \\ l_2^* = \frac{5}{4} \end{cases}$.

Remark 2. There are two forces under this setting. First, the tendency to cater to consumers brings them together (called "market size effect"). Second, the incentive to enjoy market power drives them differentiate (called "market power effect"). In fact, as pointed out in Remark 1, Question (1) captures the pure "market size effect" since there will be no "market power effect" due to fixed price.

Suggested Solutions for the Second Midterm*

Xincheng Qiu†
Peking University

May 2017

Question 1. *Bargaining with Partial Commitment*

Consider the infinite-horizon alternating-offer bargaining model discussed in class. Two agents $i = 1, 2$ bargain about how to divide a pie. Time is discrete and indexed by $t = 1, 2, \cdots$. The agents take turns to make proposals: player 1 at odd t while player 2 at even t.

Now suppose before the bargaining starts, player 2 can choose a number $z \in \left[\frac{\delta}{1+\delta}, 1\right]$. The interpretation is that player 2 takes "actions" which partially commit her not to accept a share strictly less than z. If proposal $(x, 1-x)$ is accepted at time t, 1's payoff is $\delta^{t-1}x$ and 2's payoff is $\delta^{t-1}\left[(1-x) - C(1-x, z)\right]$, where δ is the common discount factor and $C(1-x, z) = \max\{z - (1-x), 0\}$. So player 2 will suffer a psychological loss $C(1-x, z)$ is she receives a share strictly less than z.

Inspired by the subgame perfect equilibrium in the bargaining model, we consider the following strategy profile: (1) when player 1 proposes, her proposal is $(x_1, 1-x_1)$ and player 2 accepts any offer if player 1 demands less than x_1; and (2) when player 2 proposes, her proposal is $(1-x_2, x_2)$ and player 1 accepts any offer if player 2 demands less than x_2.

1. For a given z, solve $x_1$and x_2 such that the strategies described above constitute a subgame perfect equilibrium. (**Hint**: You should consider two cases: $x_2 > z$ and $x_2 \leq z$.)

2. If player 2 is free to choose any $z \in \left[\frac{\delta}{1+\delta}, 1\right]$, what will be the optimal z chosen by player 2?

Solution. 1. Let M_i be the supremum and m_i be the infimum of player i's SPE payoffs when it is player i's turn to make an offer.

*This problem set is designed by Prof. Xi Weng at Guanghua School of Management, Peking University.
†If you notice any errors or have any comments, please drop me a line at xincheng.qiu@gmail.com

Case 1. $x_2 > z$. Any offer $1-x_2 \geq \delta M_1$ proposed by player 2 would be accepted by player 1, hence $m_2 \geq 1-\delta M_1$. Any offer $1-x_2 < \delta m_1$ proposed by player 2 would be rejected by player 1, hence $M_2 \leq 1-\delta m_1$.

Case i. $1-x_1 > z$. Any offer $1-x_1 \geq \delta M_2$ proposed by player 1 would be accepted by player 2, hence $m_1 \geq 1-\delta M_2$. Any offer $1-x_1 < \delta m_2$ proposed by player 1 would be rejected by player 2, hence $M_1 \leq 1-\delta m_2$. Combining the four inequalities, we can obtain $M_1 = m_1 = \frac{1}{1+\delta}$ and $M_2 = m_2 = \frac{1}{1+\delta}$. Therefore, in the SPE, $x_1 = \frac{1}{1+\delta}$ and $x_2 = \frac{1}{1+\delta}$. Here we need $z < 1-x_1 = \frac{\delta}{1+\delta}$, which contradicts with $z \in \left[\frac{\delta}{1+\delta}, 1\right]$.

Case ii. $1-x_1 \leq z$. Any offer $2(1-x_1)-z \geq \delta M_2$ proposed by player 1 would be accepted by player 2, hence $m_1 \geq \frac{1}{2}(2-z-\delta M_2)$. Any offer $2(1-x_1)-z < \delta m_2$ proposed by player 1 would be rejected by player 2, hence $M_1 \leq \frac{1}{2}(2-z-\delta m_2)$. Combining the four inequalities, $M_1 = m_1 = \frac{2-z-\delta}{2-\delta^2}$ and $M_2 = m_2 = \frac{2-2\delta+\delta z}{2-\delta^2}$. Therefore, $x_1 = \frac{2-z-\delta}{2-\delta^2}$ and $x_2 = \frac{2-2\delta+\delta z}{2-\delta^2}$. Here we need $\frac{-\delta^2+z+\delta}{2-\delta^2} \leq z < \frac{2-2\delta+\delta z}{2-\delta^2}$, i.e., $\frac{\delta}{1+\delta} \leq z < \frac{2}{2+\delta}$.

Case 2. $x_2 \leq z$. Any offer $1-x_2 \geq \delta M_1$ proposed by player 2 would be accepted by player 1, hence $m_2 \geq 2(1-\delta M_1)-z$. Any offer $1-x_2 < \delta m_1$ proposed by player 2 would be rejected by player 1, hence $M_2 \leq 2(1-\delta m_1)-z$. Now $1-x_1 \leq x_2 \leq z$. From Case (ii) in Case (1), we know $m_1 \geq \frac{1}{2}(2-z-\delta M_2)$ and $M_1 \leq \frac{1}{2}(2-z-\delta m_2)$. Combining the four inequalities, $M_1 = m_1 = \frac{2-z}{2(1+\delta)}$ and $M_2 = m_2 = \frac{2-z}{1+\delta}$. Therefore, $x_1 = \frac{2-z}{2(1+\delta)}$ and $x_2 = \frac{2+z\delta}{2(1+\delta)}$. Here we need $z \geq \frac{2}{2+\delta}$.

To sum, if $\frac{\delta}{1+\delta} \leq z < \frac{2}{2+\delta}$, the parameters of the SPE strategy profile are

$$\left(x_1 = \frac{2-z-\delta}{2-\delta^2}, x_2 = \frac{2-2\delta+\delta z}{2-\delta^2}\right)$$

if $\frac{2}{2+\delta} \leq z \leq 1$, the parameters of the SPE strategy profile are

$$\left(x_1 = \frac{2-z}{2(1+\delta)}, x_2 = \frac{2+z\delta}{2(1+\delta)}\right)$$

2. If $\frac{\delta}{1+\delta} \leq z < \frac{2}{2+\delta}$, player 2's SPE payoff is $u_2^a = 2(1-x_1)-z = \delta x_2 = \delta\frac{2-2\delta+\delta z}{2-\delta^2}$, which increases as z increases. If $\frac{2}{2+\delta} \leq z \leq 1$, player 2's SPE payoff is $u_2^b = 2(1-x_1)-z = \delta(2x_2-z) = \delta\frac{2-z}{1+\delta}$, which decrease as z increases. Therefore, the optimal z chosen by player 2 is

$$z^* = \frac{2}{2+\delta}$$

Question 2. *Repeated Game*

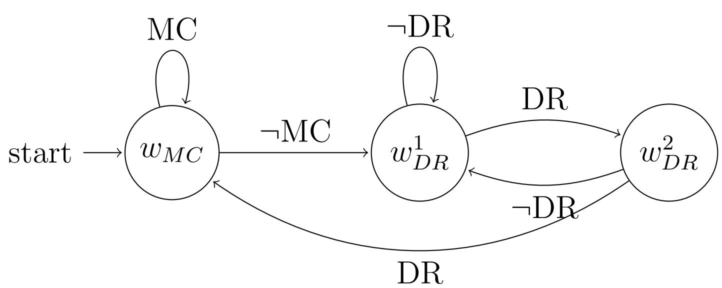
Consider the following stage game.

	L	C	R
U	5, 5	7, 0	3, 0
M	0, 0	4, x	0, 0
D	0, 0	0, 0	0, 2

1. Suppose $x = 1$ and consider the following behavior in the infinitely repeated game with perfect monitoring: Play MC in period $t = 0$. Play MC as long as no one has deviated in the previous two periods. If any player deviates, play DR for two periods and then return to MC. For what values of the common discount factor δ is this profile a subgame perfect equilibrium of the infinitely repeated game?

2. Suppose $x = 3$. How does this change to the stage game affect the range of discount factors for which the profile in (1) is a subgame perfect equilibrium of the infinitely repeated game?

Solution. The strategy profile can be represented by the following automaton:



Apply one-shot deviation principle.

At state w_{MC}, for player 1 not to deviate:

$$4 + 4\delta + 4\delta^2 + 4\delta^3 + \cdots \geq 7 + 0 + 0 + 4\delta^3 + \cdots \Rightarrow \delta \geq \frac{1}{2} \tag{1}$$

For player 2 not to deviate:

$$x + x\delta + x\delta^2 + x\delta^3 + \cdots \geq 0 + 2\delta + 2\delta^2 + x\delta^3 + \cdots \tag{2}$$

At state w^1_{DR}, player 2 will never deviate. for player 1 not to deviate:

$$0 + 0 + 4\delta^2 + 4\delta^3 + \cdots \geq 3 + 0 + 0 + 4\delta^3 + \cdots \Rightarrow \delta \geq \frac{\sqrt{3}}{2} \tag{3}$$

At state w_{DR}^2, for player 1 not to deviate:

$$0 + 4\delta + 4\delta^2 + 4\delta^3 + \cdots \geq 3 + 0 + 0 + 4\delta^3 + \cdots$$

This equation holds as long as equation 3 holds.

For player 2 not to deviate:

$$2 + x\delta + x\delta^2 + x\delta^3 + \cdots \geq 0 + 2\delta + 2\delta^2 + x\delta^3 + \cdots \tag{4}$$

1. Suppose $x = 1$. Equation 4 holds for $\forall \delta \in (0,1)$ and Equation 2 solves $\delta \leq \frac{\sqrt{5}-1}{2}$. Since $\frac{\sqrt{5}-1}{2} < \frac{\sqrt{3}}{2}$, there does not exist such δ that this profile is a SPE.

2. Suppose $x = 3$. Equation 2 and Equation 4 both hold for $\forall \delta \in (0,1)$. Therefore, for this profile to be a SPE,

$$\delta \geq \frac{\sqrt{3}}{2}$$

Question 3. *Asymmetric First Price Auction*

Consider a sealed-bid first-price auction with two buyers whose private values are independent; the private value of buyer 1 has uniform distribution over the interval $[0, 3]$, and the private value of buyer 2 has uniform distribution over the interval $[3, 4]$. Answer the following questions:

1. Derive the linear Bayesian Nash equilibrium for this game.

2. What is the probability that buyer 2 wins the auction? Why is it different from 1?

3. Compute the seller's expected revenue in the above equilibrium.

4. Suppose that the distribution of buyer 2's private value changes to uniform distribution on $[0, 4]$. Show that there does not exist a linear Bayesian Nash equilibrium.

5. Find $k_1 > 0$, $k_2 > 0$ such that the following strategies constitute a Bayesian Nash equilibrium:

$$s_1(v_1) = \frac{1}{k_1 v_1}\left(1 - \sqrt{1 - k_1 v_1^2}\right)$$
$$s_2(v_2) = \frac{1}{k_2 v_2}\left(-1 + \sqrt{1 + k_2 v_2^2}\right)$$

6. Compute the seller's expected revenue in the above equilibrium. Compare your result with the one in (3).

Solution. 1. Consider linear strategies: $s_1(v_1) = a_1 + b_1 v_1$ and $s_2(v_2) = a_2 + b_2 v_2$. For the strategy profile (s_1^*, s_2^*) to be a BNE, s_1^* maximizes

$$
\begin{aligned}
E_{v_2}[U_1 \mid v_1] &= \int_3^{\frac{s_1 - a_2}{b_2}} (v_1 - s_1)\,\mathrm{d}v_2 \\
&= (v_1 - s_1)\left(\frac{s_1 - a_2}{b_2} - 3\right)
\end{aligned}
$$

F.O.C. implies $s_1^* = \frac{1}{2}(v_1 + a_2 + 3b_2)$. Similarly, s_2^* maximizes

$$
\begin{aligned}
E_{v_1}[U_2 \mid v_2] &= \int_0^{\frac{s_2 - a_1}{b_1}} (v_2 - s_2)\,\frac{1}{3}\mathrm{d}v_1 \\
&= \frac{1}{3}(v_2 - s_2)\,\frac{s_2 - a_1}{b_1}
\end{aligned}
$$

F.O.C. implies $s_2^* = \frac{1}{2}(v_2 + a_1)$. Therefore,

$$
\begin{cases}
a_1 = \frac{1}{2}(a_2 + 3b_2) \\
b_1 = \frac{1}{2} \\
a_2 = \frac{1}{2}a_1 \\
b_1 = \frac{1}{2}
\end{cases}
\Rightarrow
\begin{cases}
a_1 = 1 \\
b_1 = \frac{1}{2} \\
a_2 = \frac{1}{2} \\
b_1 = \frac{1}{2}
\end{cases}
$$

The linear BNE is $s_1^*(v_1) = 1 + \frac{1}{2}v_1$ and $s_2^*(v_2) = \frac{1}{2} + \frac{1}{2}v_2$.[1]

2. The probability that buyer 2 wins the auction is

$$
\Pr(s_1^* < s_2^*) = \Pr(v_2 > v_1 + 1) = \frac{5}{6}
$$

3. The seller's expected revenue is

$$
\begin{aligned}
E(p) &= \int_3^4 \int_{v_2 - 1}^3 \left(1 + \frac{1}{2}v_1\right)\frac{1}{3}\mathrm{d}v_1\mathrm{d}v_2 + \int_3^4 \int_0^{v_2 - 1} \left(\frac{1}{2} + \frac{1}{2}v_2\right)\frac{1}{3}\mathrm{d}v_1\mathrm{d}v_2 \\
&= \frac{1}{3}\left(\frac{7}{6} + \frac{17}{3}\right) = \frac{41}{18}
\end{aligned}
$$

[1]To be rigorous, you need to verify that the described profile is indeed an equilibrium. This is straightforward: check the definition of BNE and verify that each player is maximizing expected payoff given the other's strategy.

4. Consider linear strategies: $s_1(v_1) = a_1 + b_1 v_1$ and $s_2(v_2) = a_2 + b_2 v_2$. For the strategy profile (s_1^*, s_2^*) to be a BNE, s_1^* maximizes

$$\begin{aligned} E_{v_2}[U_1 \mid v_1] &= \int_0^{\frac{s_1 - a_2}{b_2}} (v_1 - s_1)\,\mathrm{d}v_2 \\ &= (v_1 - s_1)\frac{s_1 - a_2}{b_2} \end{aligned}$$

F.O.C. implies $s_1^* = \frac{1}{2}(v_1 + a_2)$. Similarly, s_2^* maximizes

$$\begin{aligned} E_{v_1}[U_2 \mid v_2] &= \int_0^{\frac{s_2 - a_1}{b_1}} (v_2 - s_2)\frac{1}{3}\mathrm{d}v_1 \\ &= \frac{1}{3}(v_2 - s_2)\frac{s_2 - a_1}{b_1} \end{aligned}$$

F.O.C. implies $s_2^* = \frac{1}{2}(v_2 + a_1)$. Therefore,

$$\begin{cases} a_1 = \frac{1}{2}a_2 \\ b_1 = \frac{1}{2} \\ a_2 = \frac{1}{2}a_1 \\ b_1 = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ b_1 = \frac{1}{2} \\ a_2 = 0 \\ b_1 = \frac{1}{2} \end{cases}$$

We obtain $s_1^* = \frac{1}{2}v_1$ and $s_2^* = \frac{1}{2}v_2$. Notice that given $s_1^* = \frac{1}{2}v_1$, player 2 is sure that $s_2' = \frac{3}{2} > s_1^*$ and $s_2' < s_2^*$ when $v_2 > 3$ and thus yields a higher payoff for player 2. That is, when $v_2 > 3$, $s_2' = \frac{3}{2}$ is a better choice than $s_2^* = \frac{1}{2}v_2$, given $s_1^* = \frac{1}{2}v_1$. Therefore, $s_1^* = \frac{1}{2}v_1$ and $s_2^* = \frac{1}{2}v_2$ do not constitute a BNE.[2]

5. Given $s_2(v_2) = \frac{1}{k_2 v_2}\left(-1 + \sqrt{1 + k_2 v_2^2}\right)$, s_1^* maximizes

$$\begin{aligned} E_{v_2}[U_1 \mid v_1] &= (v_1 - s_1)\Pr(s_2 \le s_1) \\ &= \frac{1}{4}(v_1 - s_1)\frac{2s_1}{1 - k_2 s_1^2} \end{aligned}$$

F.O.C. implies

$$k_2 v_1 s_1^2 - 2s_1 + v_1 = 0$$

[2] This provides a good example illustrating why we have to verify that the candidate profile is in fact an equilibrium, as mentioned in Footnote 1.

Note that S.O.C. should also be satisfied, so

$$s_1^* = \frac{1-\sqrt{1-k_2 v_1^2}}{k_2 v_1}$$

Similarly, given $s_1(v_1) = \frac{1}{k_1 v_1}\left(1-\sqrt{1-k_1 v_1^2}\right)$, s_2^* maximizes

$$\begin{aligned} E_{v_1}[U_2 \mid v_2] &= (v_2 - s_2)\Pr(s_1 \le s_2) \\ &= \frac{1}{3}(v_2 - s_2)\frac{2s_2}{1+k_1 s_2^2} \end{aligned}$$

F.O.C. implies

$$-k_1 v_2 s_1^2 - 2s_2 + v_2 = 0$$

Combining with S.O.C. yields

$$s_2^* = \frac{-1+\sqrt{1+k_1 v_2^2}}{k_1 v_2}$$

Hence $k_1 = k_2 \equiv k$. Note that

$$\begin{aligned} \frac{\mathrm{d}s_1^*(v_1)}{\mathrm{d}v_1} &= \frac{1}{kv_1^2}\left[\frac{kv_1^2}{\sqrt{1-kv_1^2}} - \left(1-\sqrt{1-kv_1^2}\right)\right] \\ &= \frac{1}{kv_1^2}\left(\frac{1}{\sqrt{1-kv_1^2}} - 1\right) > 0 \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}s_2^*(v_2)}{\mathrm{d}v_2} &= \frac{1}{kv_2^2}\left[\frac{kv_2^2}{\sqrt{1+kv_2^2}} - \left(-1+\sqrt{1+kv_2^2}\right)\right] \\ &= \frac{1}{kv_2^2}\left(1 - \frac{1}{\sqrt{1+kv_2^2}}\right) > 0 \end{aligned}$$

$s_1^*(v_1)$ increases as v_1 increases, and $s_2^*(v_2)$ is also increasing in v_2. Now we show that $s_1^*(v_1 = 3) = s_2^*(v_2 = 4)$ by contradiction. Without loss of generality, assume $s_1^*(v_1 = 3) < s_2^*(v_2 = 4)$. The monotonicity of $s_2^*(v_2)$ guarantees that there exists some $v_0 < 4$ such that $s_2^*(v_0) = s_1^*(3)$. Thus $s_2^*(v_0) > s_1^*(v_1)$, for $\forall v_1 < 3$. Given player 1's strategy $s_1^*(v_1)$, $s_2^*(v_0)$ is a better choice for player 2 than $s_2^*(v_2)$ when $v_2 > v_0$, since player 2 will win at a lower cost. Therefore, if $s_1^*(v_1 = 3) < s_2^*(v_2 = 4)$, this profile cannot be a BNE. Similarly, we can argue that if $s_1^*(v_1 = 3) > s_2^*(v_2 = 4)$, this profile is not a BNE, either. The only possibility is $s_1^*(v_1 = 3) = s_2^*(v_2 = 4)$, i.e.,

$$\frac{1}{3k}\left(1 - \sqrt{1-9k}\right) = \frac{1}{4k}\left(-1 + \sqrt{1+16k}\right)$$

It gives $k = \frac{7}{144}$.

6. $s_1^*(v_1) \leq s_2^*(v_2)$ is equivalent to

$$v_1 \leq \frac{v_2}{\sqrt{1+kv_2^2}}$$

or

$$v_2 \geq \frac{v_1}{\sqrt{1-kv_1^2}}$$

The seller's expected revenue is

$$\begin{aligned}
E(p) &= \frac{1}{12}\left[\int_0^4 \int_0^{\frac{v_2}{\sqrt{1+kv_2^2}}} \frac{1}{kv_2}\left(-1+\sqrt{1+kv_2^2}\right) \mathrm{d}v_1 \mathrm{d}v_2 + \int_0^3 \int_0^{\frac{v_1}{\sqrt{1-kv_1^2}}} \frac{1}{kv_1}\left(1-\sqrt{1-kv_1^2}\right) \mathrm{d}v_2 \mathrm{d}v_1\right] \\
&= \frac{1}{12k}\left[\int_0^4 \left(1 - \frac{1}{\sqrt{1+kv_2^2}}\right) \mathrm{d}v_2 + \int_0^3 \left(\frac{1}{\sqrt{1-kv_1^2}} - 1\right) \mathrm{d}v_1\right] \\
&= \frac{1}{12k}\left[1 - \int_0^4 \frac{1}{\sqrt{1+kv_2^2}} \mathrm{d}v_2 + \int_0^3 \frac{1}{\sqrt{1-kv_1^2}} \mathrm{d}v_1\right] \\
&= \frac{1}{12k}\left[1 - \frac{1}{\sqrt{k}} \ln\left(\sqrt{k}v_2 + \sqrt{1+kv_2^2}\right) |_0^4 + \frac{1}{\sqrt{k}} \arcsin\left(\sqrt{k}v_1\right) |_0^3\right] \\
&= \frac{1}{12k}\left[1 - \frac{1}{\sqrt{k}} \ln\left(4\sqrt{k} + \sqrt{1+16k}\right) + \frac{1}{\sqrt{k}} \arcsin\left(3\sqrt{k}\right)\right] \\
&\approx 1.15 < \frac{41}{18}
\end{aligned}$$

Another way to calculate the expected revenue of the seller is to first derive the distribution. The cumulative distribution function of the revenue p is

$$\begin{aligned}
F(p) &= \Pr(\max\{s_1, s_2\} \leq p) = \Pr\left(v_1 \leq \frac{2p}{1+kp^2}\right) \Pr\left(v_2 \leq \frac{2p}{1-kp^2}\right) \\
&= \frac{1}{12} \frac{4p^2}{(1-k^2p^4)}
\end{aligned}$$

Note that both $s_1^*(\cdot)$ and $s_2^*(\cdot)$ are increasing in the argument, and $s_1^*(3) = s_2^*(4) = \frac{12}{7}$, which will be the upper bound for the following integral. Therefore the seller's expected revenue is[3]

$$\begin{aligned} E(p) &= \int p\mathrm{d}F(p) = \frac{1}{3}\int_0^{\frac{12}{7}} p\mathrm{d}\frac{p^2}{(1-k^2p^4)} \\ &= \frac{1}{3}\left\{p\frac{p^2}{(1-k^2p^4)}\Big|_0^{\frac{12}{7}} - \int_0^{\frac{12}{7}} \frac{p^2}{(1-k^2p^4)}\mathrm{d}p\right\} \\ &\approx 1.15 < \frac{41}{18} \end{aligned}$$

[3] In grading, it is fine that you did not get the final number for your answer. I will not deduct any point as long as you clearly write about how to proceed with the expected revenue.

Suggested Solutions for Game Theory Final 2017*

Xincheng Qiu†
Peking University

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Question 1. *Quantity Signaling.*

There are two firms, 1 and 2 producing the same good. The inverse demand curve is given by $P = \theta - q_1 - q_2$*, where* $q_i \in \mathbb{R}_+$ *is firm i's output. (Note that we are allowing negative prices.) There is demand uncertainty with nature determining the value of* θ*, assigning probability* $\alpha \in (0,1)$ *to* $\theta = 3$*, and complementary probability* $1-\alpha$ *to* $\theta = 4$*. Firm 2 is informed of the value of* θ*, while firm 1 is not. Finally, each firm has zero costs of production. As usual, assume this description is common knowledge.*

1. *Suppose that firm 2 is a Stackelberg leader. There is a separating perfect Bayesian equilibrium in which firm 2 chooses* $q_2 = \frac{1}{2}$ *when* $\theta = 3$*. Describe it, and prove it is a separating perfect Bayesian equilibrium.*

2. *Does the equilibrium in question 1 pass the intuitive criterion? Why or why not? If not, describe a separating perfect Bayesian equilibrium that does (you do not need to prove that the new equilibrium passes the intuitive criterion).*

3. *Now suppose that firm 2 is a Stackelberg leader who has the option of not choosing before firm 1: Firm 2 either chooses its quantity,* q_2*, first, or the action W (for wait). If firm 2 chooses W, then the two firms simultaneously choose quantities, knowing that they are doing so. If firm 2 chooses its quantity first (so that it did not choose W), then firm 1, knowing firm 2's quantity choice then chooses its quantity. Describe a strategy profile for this dynamic game. What conditions must a (weak) perfect Bayesian equilibrium satisfy? (Hint: You should explain explicitly what sequential rationality and belief consistency mean in this situation.)*

*This problem set is designed by Prof. Xi Weng at Guanghua School of Management, Peking University.

†If you notice any errors or have any comments, please drop me a line at xincheng.qiu@gmail.com

4. For which parameter values is there an equilibrium in which firm 2 waits for all values of θ.

Solution.

1. Construct a separating PBE in which the strategies are

$$q_2(\theta) = \begin{cases} \frac{1}{2} & \theta = 3 \\ 2 & \theta = 4 \end{cases}$$

$$q_1(q_2) = \begin{cases} \frac{1}{2}(3 - q_2) = \frac{5}{4} & q_2 = \frac{1}{2} \\ \frac{1}{2}(4 - q_2) & q_2 \neq \frac{1}{2} \end{cases}$$

and beliefs are

$$\mu(\theta = 3 \mid q_2) = \begin{cases} 1 & q_2 = \frac{1}{2} \\ 0 & q_2 \neq \frac{1}{2} \end{cases}$$

A useful notation is to denote firm 2's payoff by $U\left(\theta, \hat{\theta}, q_2\right)$, where q_2 is firm 2's choice and $\hat{\theta}$ is firm 1's belief about θ. Ignore the non-negative constraint on quantity at this moment,[1]

$$U\left(\theta, \hat{\theta}, q_2\right) = \left[\theta - \frac{1}{2}\left(\hat{\theta} - q_2\right) - q_2\right] q_2 = \left(\theta - \frac{1}{2}\hat{\theta} - \frac{1}{2}q_2\right) q_2$$

To verify the this is indeed a PBE, we need to check:

(1) It is immediate that the beliefs on the path-of-play satisfies Bayes' updating and firm 1 is playing his best response given his belief both on and off the path-of play.

(2) Firm 2's incentive compatibility constraint:

$$U\left(3, 3, \frac{1}{2}\right) = \frac{5}{8} > U(3, 4, 2) = 0$$

$$U(4, 4, 2) = 2 > U\left(4, 3, \frac{1}{2}\right) = \frac{9}{8}$$

(3) Firm 2 has no incentive to choose any quantity off the path-of-play:

$$\max_{q'} U(3, 4, q') = \frac{1}{2} < \frac{5}{8} = U\left(3, 3, \frac{1}{2}\right)$$

[1] To capture an important intuition, note that $U\left(\theta, \hat{\theta}, q_2\right)$ is decreasing in $\hat{\theta}$ in this environment. To firm 2, the "worst" belief of firm 1 takes the highest possible level of $\hat{\theta}$. In a signalling game, this corresponds to the type who must choose the full information optimal choice.

$$\max_{q'} U(4,4,q') = 2 = U(4,4,2)$$

2. It does not pass the intuitive criterion. Consider any off path quantity $q_2 \in \left(\frac{1}{2}, 1\right)$. Note that

$$U(4,3,q_2) = \left(4 - \frac{1}{2}(3-q_2) - q_2\right) q_2 < 2 = U(4,4,2)$$

Moreover,

$$U(3,3,q_2) = \left(3 - \frac{1}{2}(3-q_2) - q_2\right) q_2 > \frac{5}{8} = U\left(3,3,\frac{1}{2}\right)$$

So the dominated set $D(q_2) = \{4\}$. The above equilibrium contradicts with the intuitive criterion. The only equilibrium outcome that passes the intuitive criterion is Riley outcome, in which the full information type is indifferent between his optimal choice and imitating the other type.

$$U(4,4,2) = U(4,3,q) \implies q = 1$$

So the Riley quantities for firm 2 are given by

$$q_2(\theta) = \begin{cases} 1 & \theta = 3 \\ 2 & \theta = 4 \end{cases}$$

3. A strategy for firm 2 is a pair of functions $\{\sigma_2^1, \sigma_2^2\}$ such that $\sigma_2^1\colon \{3,4\} \to \mathbb{R}_+ \cup \{W\}$ and $\sigma_2^2\colon \{3,4\} \times \{W\} \to \mathbb{R}_+$. A strategy for firm 1 is a function $\sigma_1\colon \{W\} \cup \mathbb{R}_+ \to \mathbb{R}_+$.[2] A strategy profile is a (weak) perfect Bayesian equilibrium if there exist a system beliefs satisfying the following:

(1) The beliefs satisfy Bayes' rule on the path of play.

(2) After the choice W by firm 2, the players' choices constitute a Nash equilibrium of the resulting incomplete information game given the players' beliefs.

(3) After any choice q_2 other than W by firm 2, firm 1's choice must be sequentially rational given its beliefs.

(4) Firm 2's strategy σ_2^1 must maximize its profits given σ_1.

4. Suppose we are now in such an equilibrium. Consider the "worst" beliefs: firm 1 will believe $\theta = 4$ with probability 1 if he observes any behavior by firm 1 other than W. From

[2] $\sigma_2^1(\theta)$ describes 2's choice about whether or not to wait, and if not what quantity. σ_2^2 specifies 2's choice of quantity after waiting. $\sigma_1(W)$ means 1's quantity choice if 2 waits and $\sigma_1(q)$ if 2 does not wait and chooses q.

previous question, we know that if firm 2 deviates he will get the payoff of 2. To exclude profitable deviation,

$$\left(\frac{8+\alpha}{6}\right)^2 \geq 2$$

needs to hold. It solves for

$$6\sqrt{2}-8 \leq \alpha \leq 1$$

Question 2. *Selling a Firm.*

The owner of a small firm is contemplating selling all or part of his firm to outside investors. The profits from the firm are risky and the owner is risk averse. The owner's preferences over x, the fraction of the firm the owner retains, and p, the price "per share" paid by the outside investors, are given by

$$u(x,\theta,p) = \theta x - x^2 + p(1-x)$$

where $\theta > 1$ is the value of the firm (i.e., expected profits). The quadratic term reflects the owner's risk aversion. The outside investors are risk neutral, and so the payoff to an outside investor of paying p per share for $1-x$ of the firm is then

$$\theta(1-x) - p(1-x)$$

There are at least two outside investors, and the price is determined by a first price sealed bid auction: The owner first chooses the fraction of the firm to sell, $1-x$; the outside investors then bid, with the $1-x$ fraction going to the highest bidder (ties are broken with a coin flip).

1. *Suppose θ is public information. What fraction of the firm will the owner sell, and how much will he receive for it?*

2. *Suppose θ is privately known by the owner. The outside investors have common beliefs, assigning probability α to $\theta = \theta_1$ and probability $1-\alpha$ to $\theta = \theta_2 > \theta_1$. Suppose $\theta_2 - \theta_1 > 2$. Characterize one separating perfect Bayesian equilibrium of this game.*

3. *Now, instead of assuming that there are at least two investors, suppose that there is only one investor. As a monopolist, this investor can offer a menu contract $\{p_1, x_1; p_2, x_2\}$ to maximize her expected profits. The outside option of the owner is such that the owner can fully retain the firm ($x = 1$), and get utility $\theta - 1$. Solve the optimal screening contract proposed by the investor.*

Solution.

1. In a first sealed bid auction under public information, $p = \theta$. Then the owner's problem becomes

$$\max_{x} u(x, \theta, p) = \theta x - x^2 + \theta(1 - x)$$

F.O.C. implies that $x = 0$. The owner will sell all of the firm and receive θ for it.

2. In the separating PBE, the owner chooses $x = x_1$ when observing $\theta = \theta_1$ and chooses $x = x_2$ when observing $\theta = \theta_2$. The outside investor's belief is $\mu(\theta = \theta_1 \mid x = x_1) = 1$ and $\mu(\theta = \theta_1 \mid x = x_2) = 0$. Given this belief, the investor will choose $p_1 = \theta_1$ when observing x_1 and $p_2 = \theta_2$ when observing x_2. Since θ_1 is already the lowest possible price, type θ_1 will choose the full information $x_1^* = 0$. The incentive compatibility constraint for θ_1 is

$$\theta_1 \geq \theta_1 x_2 - x_2^2 + \theta_2(1 - x_2) \tag{1}$$

This implies

$$x_2^* \geq \frac{-(\theta_2 - \theta_1) + \sqrt{(\theta_2 - \theta_1)^2 + 4(\theta_2 - \theta_1)}}{2} \equiv \bar{x}$$

The incentive compatibility constraint for θ_2 is

$$\theta_2 x_2 - x_2^2 + \theta_2(1 - x_2) \geq \theta_1 \tag{2}$$

which requires $x_2^2 \leq \theta_2 - \theta_1$. This trivially hold since $x_2 \leq 1$ and $\theta_2 - \theta_1 > 2$. Specify the worst belief for off path quantities and thus $p(x) = \theta_1$ for $x \neq 0, x_2^*$. We need

$$\theta_2 x_2 - x_2^2 + \theta_2(1 - x_2) \geq \max_{x} \theta_2 x - x^2 + \theta_1(1 - x)$$

The solution to the RHS maximization problem is $x^* = \min\left\{1, \frac{\theta_2 - \theta_1}{2}\right\}$. Suppose $\theta_2 - \theta_1 > 2$. $x^* = 1$ and we need to make sure that

$$\theta_2 x_2 - x_2^2 + \theta_2(1 - x_2) \geq \theta_2 - 1 \tag{3}$$

Thus any $x_2 \in [\bar{x}, 1]$ is consistent with a separating PBE. There are also pooling equilibria, where both types choose the same x_p, with $p(x_p) = \alpha\theta_1 + (1 - \alpha)\theta_2 \equiv \bar{\theta}$. Again specify the worst belief for off path quantities such that $p(x) = \theta_1$ for $x \neq x_p$. We need

$$\theta_1 x_p - x_p^2 + \bar{\theta}(1 - x_p) \geq \theta_1$$

3. The optimization problem for the owner is

$$\max_{\{p_1,x_1;p_2,x_2\}} \alpha\left[\theta_1\left(1-x_1\right)-p_1\left(1-x_1\right)\right]+\left(1-\alpha\right)\left[\theta_2\left(1-x_2\right)-p_2\left(1-x_2\right)\right]$$

$$\text{s.t.}\begin{cases}\theta_1 x_1-x_1^2+p_1\left(1-x_1\right)\geq\theta_1-1 & IR_1\\ \theta_2 x_2-x_2^2+p_2\left(1-x_2\right)\geq\theta_2-1 & IR_2\\ \theta_1 x_1-x_1^2+p_1\left(1-x_1\right)\geq\theta_1 x_2-x_2^2+p_2\left(1-x_2\right) & IC_1\\ \theta_2 x_2-x_2^2+p_2\left(1-x_2\right)\geq\theta_2 x_1-x_1^2+p_1\left(1-x_1\right) & IC_2\end{cases}$$

IR_1 and IC_2 could be redundant. Substituting IR_2 and IC_1 into the objective function yields

$$\max_{\{p_1,x_1;p_2,x_2\}} \alpha\left[\theta_1-x_1^2-\theta_1 x_2+1-\theta_2\left(1-x_2\right)\right]+\left(1-\alpha\right)\left[1-x_2^2\right]$$

F.O.C. implies

$$\begin{cases}x_1^*=0\\ x_2^*=\frac{\alpha(\theta_2-\theta_1)}{2(1-\alpha)}\end{cases}$$

And

$$\begin{cases}p_1^*=-\frac{\alpha(\theta_2-\theta_1)^2}{2(1-\alpha)}+\theta_2-1\\ p_2^*=-\frac{\alpha(\theta_2-\theta_1)}{2(1-\alpha)}+\theta_2-1\end{cases}$$

Question 3. *Monopoly Screening.*

Consider the following optimal pricing (screening) problem with quality-differentiated products. There is a continuum of consumers whose preferences are given by $u=\theta v\left(q\right)-t\left(q\right)$ *where*

$$v\left(q\right)=\frac{1-\left(1-q\right)^2}{2}$$

The proportion of consumers with high valuation θ_h *is given by* λ *and the proportion of consumers with low valuation* θ_l *is given by* $1-\lambda$*. The monopolist has a constant marginal cost equal to* c *of producing quality* q*, with* $0<c<\theta_l<\theta_h$*. If a consumer does not purchase, she receives an outside option of* 0*.*

1. *Derive the monopolist's optimal menu subject to the participation constraint of the buyers, assuming for the moment that he can observe the type* θ *of the buyer.*

2. *Suppose now that θ is not observable. Derive the monopolist's optimal solution. Carefully specify conditions under which it is indeed optimal to choose both q_h and q_l to be strictly positive.*

3. *Calculate the monopolist's expected profits under both situations described in 1 and 2. How does the reduction in expected profits due to asymmetric information change with c and λ? Explain the intuition.*

Solution.

1. Assume that the monopolist can observe the type of a buyer. The optimization problem becomes

$$\max_{q(\theta),t(\theta)} t - cq$$

$$\text{s.t. } \theta v(q) - t \geq 0$$

The optimal menu is

$$(q,t) = \left(1 - \frac{c}{\theta}, \frac{\theta^2 - c^2}{2\theta}\right)$$

2. Assume that the monopolist cannot observe the type of a buyer. The optimization problem becomes

$$\max_{q_h,t_h,q_l,t_l} \lambda(t_h - cq_h) + (1-\lambda)(t_l - cq_l)$$

$$\text{s.t.} \begin{cases} \theta_h v(q_h) - t_h \geq 0 & IR_h \\ \theta_l v(q_l) - t_l \geq 0 & IR_l \\ \theta_h v(q_h) - t_h \geq \theta_h v(q_l) - t_l & IC_h \\ \theta_l v(q_l) - t_l \geq \theta_l v(q_h) - t_h & IC_l \end{cases}$$

The buyers with high valuation have incentive to mimic those with low valuation. IR_l and IC_h are binding while the other two are redundant. So the optimization problem can be simplified as

$$\max_{q_h,t_h,q_l,t_l} \lambda(t_h - cq_h) + (1-\lambda)(t_l - cq_l)$$

$$\text{s.t.} \begin{cases} \theta_l v(q_l) - t_l = 0 & IR_l \\ \theta_h v(q_h) - t_h = \theta_h v(q_l) - t_l & IC_h \end{cases}$$

The solution is given by

$$\begin{cases} q_h^* = 1 - \frac{c}{\theta_h} \\ q_l^* = 1 - \frac{(1-\lambda)c}{\theta_l - \lambda\theta_h} \end{cases}$$

To guarantee that $q_l^* > 0$, we need $\lambda < \frac{\theta_l - c}{\theta_h - c}$. Under this condition,

$$\begin{cases} t_h^* = \frac{\theta_h}{2}\left(1 - \frac{c^2}{\theta_h^2}\right) - \frac{\theta_h - \theta_l}{2}\left[1 - \frac{(1-\lambda)^2 c^2}{(\theta_l - \lambda\theta_h)^2}\right] \\ t_l^* = \frac{\theta_l}{2}\left[1 - \frac{(1-\lambda)^2 c^2}{(\theta_l - \lambda\theta_h)^2}\right] \end{cases}$$

3. The expected profits under situation 1 is

$$\begin{aligned} E\pi_1 &= \lambda\left(\frac{\theta_h^2 - c^2}{2\theta_h} - c\left(1 - \frac{c}{\theta_h}\right)\right) + (1-\lambda)\left(\frac{\theta_l^2 - c^2}{2\theta_l} - c\left(1 - \frac{c}{\theta_l}\right)\right) \\ &= \lambda\frac{(\theta_h - c)^2}{2\theta_h} + (1-\lambda)\frac{(\theta_l - c)^2}{2\theta_l} \end{aligned}$$

The expected profits under situation 2 is

$$\begin{aligned} E\pi_2 &= \lambda\left(t_h^* - cq_h^*\right) + (1-\lambda)\left(t_l^* - cq_l^*\right) \\ &= \lambda\left(\frac{(\theta_h - c)^2}{2\theta_h} - \frac{\theta_h - \theta_l}{2}\left[1 - \frac{(1-\lambda)^2 c^2}{(\theta_l - \lambda\theta_h)^2}\right]\right) + (1-\lambda)\left(\frac{\theta_l}{2}\left[1 - \frac{(1-\lambda)^2 c^2}{(\theta_l - \lambda\theta_h)^2}\right] - c\left[1 - \frac{(1-\lambda)c}{\theta_l - \lambda\theta_h}\right]\right) \end{aligned}$$

The reduction

$$\begin{aligned} \Delta &= E\pi_1 - E\pi_2 \\ &= \lambda\frac{\theta_h - \theta_l}{2}\left[1 - \frac{(1-\lambda)^2 c^2}{(\theta_l - \lambda\theta_h)^2}\right] + (1-\lambda)\left[\frac{c^2}{2\theta_l} + \frac{(1-\lambda)^2 c^2\theta_l}{2(\theta_l - \lambda\theta_h)^2} - \frac{(1-\lambda)c^2}{\theta_l - \lambda\theta_h}\right] \\ &= \frac{1}{2(\theta_l - \lambda\theta_h)\theta_l}\left[\lambda(\theta_h - \theta_l)\theta_l(\theta_l - \lambda\theta_h) - \lambda(1-\lambda)c^2(\theta_h - \theta_l)\right] \\ &= \frac{\lambda(\theta_h - \theta_l)}{2(\theta_l - \lambda\theta_h)\theta_l}\left[\theta_l(\theta_l - \lambda\theta_h) - (1-\lambda)c^2\right] \\ &= \frac{\theta_h - \theta_l}{2}\lambda\left[1 + \frac{(1-\lambda)c^2}{(\lambda\theta_h - \theta_l)\theta_l}\right] \end{aligned}$$

Since $\lambda < \frac{\theta_l - c}{\theta_h - c}$,

$$\begin{aligned}\lambda\theta_h - \theta_l &< \frac{\theta_l - c}{\theta_h - c}\theta_h - \theta_l \\ &= \frac{1}{\theta_h - c}\left(\left(\theta_l - c\right)\theta_h - \left(\theta_h - c\right)\theta_l\right) \\ &= \frac{-c}{\theta_h - c}\left(\theta_h - \theta_l\right) < 0\end{aligned}$$

Thus

$$\frac{\partial \Delta}{\partial c} = \left(\theta_h - \theta_l\right)\frac{\left(1-\lambda\right)\lambda}{\left(\lambda\theta_h - \theta_l\right)\theta_l}c < 0$$

And

$$\begin{aligned}\frac{\partial \Delta}{\partial \lambda} &= \frac{\theta_h - \theta_l}{2}\left(1 + \frac{c^2}{\theta_l}\frac{\left(1-2\lambda\right)\left(\lambda\theta_h - \theta_l\right) - \left(\lambda - \lambda^2\right)\theta_h}{\left(\lambda\theta_h - \theta_l\right)^2}\right) \\ &= \frac{\theta_h - \theta_l}{2}\left[1 + \frac{c^2}{\theta_l}\frac{\left(-\theta_l + 2\lambda\theta_l - \lambda^2\theta_h\right)}{\left(\lambda\theta_h - \theta_l\right)^2}\right] \\ &= \frac{\theta_h - \theta_l}{2\left(\lambda\theta_h - \theta_l\right)^2\theta_l}\left[\left(\lambda\theta_h - \theta_l\right)^2\theta_l + c^2\left(\lambda - 1\right)\theta_l - c^2\lambda\left(\lambda\theta_h - \theta_l\right)\right]\end{aligned}$$

The sign of $\frac{\partial \Delta}{\partial \lambda}$ depends on $A \triangleq \left(\lambda\theta_h - \theta_l\right)^2\theta_l + c^2\left(\lambda - 1\right)\theta_l - c^2\lambda\left(\lambda\theta_h - \theta_l\right)$. Note that

$$\begin{aligned}\frac{\partial A}{\partial \lambda} &= 2\left(\lambda\theta_h - \theta_l\right)\theta_h\theta_l + 2c^2\left(\theta_l - \theta_h\lambda\right) \\ &= 2\left(\lambda\theta_h - \theta_l\right)\left(\theta_h\theta_l - c^2\right) < 0\end{aligned}$$

and

$$A \mid_{\lambda=0} = \theta_l^3 - c^2\theta_l = \left(\theta_l^2 - c^2\right)\theta_l > 0$$

$$
\begin{aligned}
A\mid_{\lambda=\frac{\theta_l-c}{\theta_h-c}} &= \left(\lambda\theta_h-\theta_l\right)^2\theta_l + c^2\left(\lambda-1\right)\theta_l - c^2\lambda\left(\lambda\theta_h-\theta_l\right) \\
&= \left[\left(\lambda\theta_h-\theta_l\right)\theta_l - c^2\lambda\right]\left(\lambda\theta_h-\theta_l\right) + c^2\left(\lambda-1\right)\theta_l \\
&= \left[\left(\frac{\theta_l-c}{\theta_h-c}\theta_h-\theta_l\right)\theta_l - c^2\frac{\theta_l-c}{\theta_h-c}\right]\left(\frac{\theta_l-c}{\theta_h-c}\theta_h-\theta_l\right) + c^2\left(\frac{\theta_l-c}{\theta_h-c}-1\right)\theta_l \\
&= \frac{c^2}{\left(\theta_h-c\right)^2}\left[\left(\theta_h-\theta_l\right)\theta_l + c\left(\theta_l-c\right)\right]\left(\theta_h-\theta_l\right) + c^2\left(\frac{\theta_l-c}{\theta_h-c}-1\right)\theta_l \\
&= -\frac{c^2\left(\theta_h-\theta_l\right)}{\left(\theta_h-c\right)^2}\left(\theta_l^2+c^2\right) < 0
\end{aligned}
$$

So there exists some $\bar{\lambda} \in \left(0, \frac{\theta_l-c}{\theta_h-c}\right)$ such that when $0 < \lambda < \bar{\lambda}$, $\frac{\partial\Delta}{\partial\lambda} > 0$, and when $\bar{\lambda} < \lambda < \frac{\theta_l-c}{\theta_h-c}$, $\frac{\partial\Delta}{\partial\lambda} < 0$.

Intuition: When c becomes larger, the profit becomes smaller, so the value of information becomes smaller. As $\lambda \to 0$ or $\lambda \to 1$, it approximates to the case with complete information. So when λ gets larger, the effect of asymmetric information (reduction in expected profits) first becomes more severe and then less severe.