Penn Econ Math Camp Part II

Final Exam

Instructor: Xincheng Qiu

August 23, 2021

This part has 40 points. Indicate your reasoning and write legibly. If you skip a subquestion, you can continue to the next one assuming that the result in the previous one holds.

1. (18 points) Consider a stochastic matrix (entries are non-negative and the sum of the entries in each column adds up to one)

$$P = \begin{pmatrix} 1-f & s \\ f & 1-s \end{pmatrix}.$$

1. (4 points) Under what conditions for f and s is P a projection matrix?

Solution: P is a projection matrix if and only if P is idempotent and symmetric. For P to be symmetric, we need $P = P'$, that is, $s = f$. For P to be idempotent, we need $P = PP$, which could be written as

$$\begin{pmatrix} 1-f & f \\ f & 1-f \end{pmatrix} = \begin{pmatrix} (1-f)^2 + f^2 & 2f(1-f) \\ 2f(1-f) & (1-f)^2 + f^2 \end{pmatrix}$$

i.e.

$$\begin{cases} 1 - f = (1-f)^2 + f^2 \\ f = 2f(1-f) \end{cases}$$

which solves for $f = s = 0$ or $f = s = 1/2$.

2. (3 points) Under what conditions for f and s is P^tP full rank?

Solution: P^tP is full rank if and only if P full rank. P is full rank if and only if $det(P) \neq 0$, where

$$det(P) = (1-f)(1-s) - sf = 1 - (f+s).$$

That is, P full rank if and only if $f + s \neq 1$.

3. (3 points) Under what conditions for f and s is P positive definite?

Solution: First of all, P needs to be symmetric, i.e., $s = f$.

Second, P is positive definite if and only if it has a Cholesky decomposition, i.e., $P = LL^t$ for some lower triangular full rank matrix L. Denote

$$L = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix},$$

thus $P = LL^t$ requires

$$\begin{pmatrix} 1-f & f \\ f & 1-f \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a^2 & ab \\ ab & b^2 + c^2 \end{pmatrix}.$$

Let $a = \sqrt{1-f}, b = \frac{f}{\sqrt{1-f}}, c = \frac{\sqrt{1-2f}}{\sqrt{1-f}}$, where we need $f \leq 1/2$ for c to be real. The matrix L is full rank if and only if $det(L) \neq 0$, where

$$det\,(L) = ac = \sqrt{1-2f} \neq 0.$$

In sum, we need $s = f < 1/2$.

Alternative Solution: By definition, P is positive definite if $x^t P x > 0$ for any $x \neq 0$. Take an arbitrary nonzero vector $x = (x_1, x_2)$. Then

$$x^t P x = (x_1, x_2) \begin{pmatrix} 1-f & f \\ f & 1-f \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (1-f)\,x_1^2 + 2f x_1 x_2 + (1-f)\,x_2^2$$

If $x_2 = 0$, then $x_1 \neq 0$, and $x^t P x = (1-f)\,x_1^2 > 0$ as long as $1 - f > 0$. If $x_2 \neq 0$, then we can write

$$\begin{aligned} x^t P x &= (1-f)\,x_2^2 \left[\left(\frac{x_1}{x_2}\right)^2 + \frac{2f}{1-f}\frac{x_1}{x_2} + 1 \right] \\ &= (1-f)\,x_2^2 \left[\left(\frac{x_1}{x_2} + \frac{f}{1-f}\right)^2 + 1 - \left(\frac{f}{1-f}\right)^2 \right]. \end{aligned}$$

For it to be always strictly greater than 0, we need $1 - \left(\frac{f}{1-f}\right)^2 > 0$, i.e., $f < 1/2$.

Alternative Solution: Another equivalent definition of a symmetric matrix being positive definite is that all its leading principal minors have positive determinants. That is, $det(P_{11}) = 1 - f > 0$ and $det(P) = (1-f)^2 - f^2 > 0$. Therefore, $s = f < 1/2$.

Alternative Solution: Another equivalent definition of a symmetric matrix being positive definite is that all its eigenvalues are positive. The characteristic polynomial of P is

$$det\left(\lambda I - P\right) = det\begin{pmatrix} \lambda - (1-f) & -f \\ -f & \lambda - (1-f) \end{pmatrix} = \left(\lambda - 1\right)\left(\lambda - (1-2f)\right),$$

so that the two eigenvalues are 1 and $1-2f$. Therefore, $s = f < 1/2$.

4. (4 points) A vector x_0 is a probability distribution if its entries are non-negative and add up to one. A probability distribution vector x^* is a stationary distribution of the stochastic matrix P if $Px^* = x^*$. Find the stationary distributions of P.

Solution: Denote x by

$$x^* = \begin{pmatrix} u \\ 1-u \end{pmatrix},$$

where $u \in [0,1]$. A stationary distribution satisfies $Px^* = x^*$, i.e.,

$$\begin{pmatrix} (1-f)u + s(1-u) \\ fu + (1-s)(1-u) \end{pmatrix} = \begin{pmatrix} u \\ 1-u \end{pmatrix}$$

which reduces to

$$(s+f)u = s.$$

(1) If $s + f > 0$, then $u^* = \frac{s}{s+f}$.
(2) Otherwise, i.e., if $s = f = 0$, the stochastic matrix becomes

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is an identity. In this case, any probability vector is a stationary distribution.

5. (4 points) Suppose all entries of P are strictly positive. Prove that, for an arbitrary probability distribution vector x_0,

$$\lim_{n\to\infty} P^n x_0 = x^*,$$

where x^* is the stationary distribution vector.

Solution: Consider a mapping $g(x) := Px$, such that $g : M \to M$, where the space of probability distribution vectors, M, is complete. We can show that g is a contraction.

To see this, consider two arbitrary probability distribution vectors, $x, y \in M$. Denote

$$x = \begin{pmatrix} u \\ 1-u \end{pmatrix}, y = \begin{pmatrix} v \\ 1-v \end{pmatrix},$$

where $u, v \in [0,1]$. Thus,

$$x - y = \begin{pmatrix} u-v \\ -(u-v) \end{pmatrix} = (u-v) \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

so that

$$d(x,y) = \|x-y\| = \sqrt{2}\,|u-v|\,.$$

Further note that,

$$\begin{aligned} Px - Py = P(x-y) &= (u-v) \begin{pmatrix} 1-f & s \\ f & 1-s \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= (u-v) \begin{pmatrix} 1-f-s \\ f-1+s \end{pmatrix} = (u-v)(1-f-s) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

so that

$$d(Px, Py) = \|Px - Py\| = P\,\|x-y\| = \sqrt{2}\,|u-v|\,|1-f-s|\,.$$

Since we assume all entries of P are strictly positive, we have $1-f \in (0,1)$ and $s \in (0,1)$, so that $-1 < 1-f-s < 1$, and hence $|1-f-s| < 1$. Therefore,

$$d(Px, Py) = \rho d(x,y)\,,$$

for $\rho := |1-f-s| < 1$. Thus g is a contraction mapping. From the contraction mapping theorem, we know that g has a unique fixed point, and by definition, x^* is its fixed point. The contraction mapping theorem guarantees that

$$\lim_{n\to\infty} P^n x_0 = x^*,$$

for any $x_0 \in M$.

Alternative Solution: Note that $x^* = \left(\frac{s}{s+f}, \frac{f}{s+f}\right)^t$ and $\tilde{x} = (1,-1)^t$ are the eigenvectors of P associated with the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1-f-s$, respectively, i.e., $Px^* = \lambda_1 x^*$ and $P\tilde{x} = \lambda_2 \tilde{x}$. Pick an arbitrary probability distribution $x_0 = (u, 1-u)^t$. Observe that we can rewrite x_0 as

$$x_0 = x^* + \left(u - \frac{s}{s+f}\right)\tilde{x}.$$

Therefore,

$$P^n x_0 = P^n x^* + \left(u - \frac{s}{s+f}\right) P^n \tilde{x} = \lambda_1^n x^* + \left(u - \frac{s}{s+f}\right) \lambda_2^n \tilde{x}$$
$$= x^* + \left(u - \frac{s}{s+f}\right) (1-f-s)^n \, \tilde{x}.$$

Since $|1-f-s| < 1$, we have $(1-f-s)^n \to 0$ as $n \to \infty$. As a result, $\lim_{n\to\infty} P^n x_0 = x^*$.

2. (17 points) Consider a risky asset, whose rate of return $\tilde{r}$ is a random variable with $n \geq 2$ possible realizations: with probability p_i, the rate of return is r_i, $\forall i = 1, 2, ..., n$. The expected return is positive, i.e.,

$$\sum_{i=1}^{n} p_i r_i > 0. \tag{1}$$

Suppose the agent's initial wealth is $w > 0$. Therefore the ex post wealth when x is invested in this asset becomes $\tilde{y} = w - x + (1+\tilde{r})x = w + \tilde{r}x$. The agent chooses $x \geq 0$ in order to maximize the expected utility

$$U(x) := \sum_{i=1}^{n} p_i u(w + r_i x), \tag{2}$$

where $u' > 0$, $u'' < 0$. Denote the optimal choice by $x^*(w)$. Define

$$A(x) = -\frac{u''(x)}{u'(x)}. \tag{3}$$

We assume $A(x)$ is strictly decreasing.

1. (3 points) Is $u' > 0$ equivalent to u being strictly increasing (assuming u differentiable)? Is $u'' < 0$ equivalent to u being strictly concave (assuming u twice differentiable)? Comment on both directions of the equivalence. Provide a counterexample if your answer is no.

Solution: If $u' > 0$, then u is strictly increasing. The reverse is not true, however. For example, $u(x) = x^3$ is strictly increasing everywhere but $u'(0) = 0$.

If $u'' < 0$, then u is strictly concave. The reverse is not true, however. For example, $u(x) = -x^4$ is strictly concave but $u''(0) = 0$.

2. (4 points) Write down the interior first order condition for $x^*(w)$. Prove that the optimal solution is indeed interior, i.e., show that $x^*(w) \neq 0$.

Solution: The interior first order condition is

$$\sum_{i=1}^{n} p_i r_i u'(w + r_i x^*(w)) = 0.$$

We will show that $U'(0) > 0$. Evaluating $U'(x)$ at $x = 0$, we have

$$U'(0) = \sum_{i=1}^{n} p_i r_i u'(w) = u'(w) \sum_{i=1}^{n} p_i r_i > 0,$$

since $u' > 0$ and $\sum_{i=1}^{n} p_i r_i > 0$. This means U is increasing at $x = 0$ and hence $x^*(w) \neq 0$.

Note: $x = 0$, if is a solution, is a corner solution. So merely stating that the interior first order condition does not hold at $x = 0$ is not sufficient to show that $x = 0$ is not a solution.

3. (4 points) Show that $\sum_{i=1}^{n} p_i r_i u''(w + r_i x^*(w)) > 0$.

Solution: From the previous question, we know that $x^*(w) > 0$.

(i) Consider $r_i > 0$. Thus $w + r_i x^*(w) > w$, which implies $A(w + r_i x^*(w)) < A(w)$ because A is strictly decreasing. Therefore $r_i A(w + r_i x^*(w)) < r_i A(w)$.

(ii) Consider $r_i < 0$. Thus $w + r_i x^*(w) < w$, which implies $A(w + r_i x^*(w)) > A(w)$ because A is strictly decreasing. Therefore $r_i A(w + r_i x^*(w)) < r_i A(w)$.

Notice that by definition of $A(\cdot)$, we could rewrite

$$\begin{aligned}
\sum_{i=1}^{n} p_i r_i u''(w + r_i x^*(w)) &= -\sum_{i=1}^{n} p_i r_i A(w + r_i x^*(w)) u'(w + r_i x^*(w)) \\
&> -\sum_{i=1}^{n} p_i r_i A(w) u'(w + r_i x^*(w)) \\
&= -A(w) \underbrace{\sum_{i=1}^{n} p_i r_i u'(w + r_i x^*(w))}_{=0 \text{ due to FOC}} = 0.
\end{aligned}$$

where the inequality is using the result as stated in the hint and the last equality holds because of the first order condition.

4. (3 points) Show that $x^*(w)$ is increasing in w.

Solution: Let $H(x, w) := \sum_{i=1}^{n} p_i r_i u'(w + r_i x)$. Consider

$$\mathbf{B} = \frac{\partial H}{\partial x}(x^*(w), w) = \sum_{i=1}^{n} p_i r_i^2 u''(w + r_i x^*(w)) < 0$$

because $p_i > 0$, $r_i^2 > 0$ for $r_i \neq 0$, $u'' < 0$. And

$$\mathbf{A} = \frac{\partial H}{\partial w}(x^*(w), w) = \sum_{i=1}^{n} p_i r_i u''(w + r_i x^*(w)) > 0$$

because of the previous result.

Apply the implicit function theorem:

$$\frac{\partial x^*(w)}{\partial w} = -\mathbf{B}^{-1}\mathbf{A} > 0,$$

which implies that $x^*(w)$ is increasing in w.

5. (3 points) Define $g = u' \circ u^{-1}$. Show that g is a convex function.

Solution: By the chain rule and the inverse function theorem,

$$g'(\cdot) = \frac{u''\left(u^{-1}(\cdot)\right)}{u'\left(u^{-1}(\cdot)\right)} = -A\left(u^{-1}(\cdot)\right)$$

Since u is strictly increasing, so is u^{-1}. Since A is decreasing, we have g' increasing, which means g is convex.

Note: We did not assume existence of u'''.

3. (5 points)

1. (1 point) State a weak separating hyperplane theorem that separates a set and a point.

Solution: Suppose $\mathcal{X} \subseteq \mathbb{R}^n$ is a non-empty convex set, $d \notin int(\mathcal{X})$. Then there exists a non-zero $p \in \mathbb{R}^n$ such that $p^t x \geq p^t d, \forall x \in \mathcal{X}$.

2. (2 point) Using the theorem stated in the previous question, show that there exists a non-zero vector $p \in \mathbb{R}^n$ such that

$$p^t x \geq 0, \forall x \geq 0,$$

where $x \in \mathbb{R}^n$.

Solution: Take $\mathcal{X} = \mathbb{R}^n_+$ and $d = 0$. We can easily verify that $\mathbb{R}^n_+$ is a non-empty convex set and $0 \notin int(\mathbb{R}^n_+)$. Thus the assumptions in the weak separating hyperplane theorem stated above hold. Therefore, there exists a non-zero $p \in \mathbb{R}^n$ such that $p^t x \geq p^t d = 0, \forall x \in \mathbb{R}^n_+$.

3. (2 point) Furthermore, prove that $p > 0$ (all coordinates are non-negative but at least one is strictly positive).

Solution: Suppose $\exists p_i < 0$. Choose x such that $x_i = 1$ and $x_j = 0, \forall j \neq i$. Then $p^t x = p_i x_i < 0$. This is a contradiction. Therefore we must have $p_i \geq 0, \forall i$. Furthermore, since $p \neq 0$, we have $p > 0$.